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MATHEMATICAL REVIEWS

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Mathematical Reviews

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HISTORY

Bruins, E. M. A contribution to the interpretation of Babylonian mathematics; triangles with regular sides. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 412–422 (1953).

Neugebauer, O. Babylonian planetary theory. Proc. Amer. Philos. Soc. 98, 60–89 (1954).

***Plooij, Edward Bernard. Euclid's conception of ratio and his definition of proportional magnitudes as criticized by Arabian commentators (including the text in facsimile with translation of the commentary on ratio of Abû 'Abd Allâh Muḥammad ibn Mu'âdh al-Djajjânî).** Thesis, University of Leiden, 1950. vi+71 pp.

This is an account of medieval Arabic discussion of two Greek definitions of proportion: the Eudoxian one given in the Elements of Euclid, and a second which has survived explicitly nowhere in the Greek literature. This latter, in effect, makes two ratios equal if and only if their continued fraction expansions, finite or infinite, are identical.

Chapter I outlines the development of Islamic mathematics and lists some forty-nine mathematicians of the period who are known to have written works on Euclid. The treatise on ratio of one of them, al-Jayyani (fl. 1080, the author transliterates the name as al-Djajjânî), is reproduced in facsimile as Chapter II, together with an opposite page translation into English, critical apparatus, and notes. It can be inferred that the manuscript used was Algiers 1446, 3°, and that this was the only copy available, although this is not stated in the publication.

The remaining chapters, III and IV, discuss in some detail the content of the treatise in conjunction with the parallel works of Thabit ibn Qurra, al-Mahani, and an-Nairizi, who state the non-Eudoxian definition explicitly; Ibn al-Haitham, who, with al-Jayyani, essays a proof of the equivalence of the two definitions; and Omar Khayyam, who criticizes Euclid's (really Eudoxus') definition while al-Jayyani upholds it. *E. S. Kennedy.*

Müller, Wilhelm. Das isoperimetrische Problem im Altertum mit einer Übersetzung der Abhandlung des Zenodoros nach Theon von Alexandrien. Sudhoffs Arch. 37, 39–71 (1953).

Clagett, Marshall. Medieval mathematics and physics: A check list of microfilm reproductions. Isis 44, 371–381 (1953).

Boas, George. A fourteenth-century cosmology. Proc. Amer. Philos. Soc. 98, 50–59 (1954).

De Lorenzo, Giuseppe. Concezioni cosmiche di Leopardi. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 179–190 (1953).

Conte, Luigi. Il Libro III della Divinazione vivianea del "De locis solidis" di Aristeo il Vecchio. Period. Mat. (4) 31, 265–274 (1953).

Cavallaro, Vincenzo G. Divina proportione. Giorn. Mat. Battaglini (5) 1(81), 203–218 (1953).

Hadamard, J. Non-Euclidian geometry and axiomatic definitions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 199–208 (1953). (Hungarian)

Philosophical and historical comments. *P. R. Halmos.*

Alexits, György. Life and activity of János Bolyai. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 131–150 (1953). (Hungarian)

Rényi, Alfréd. The significance of the viewpoint of the geometry of Bolyai-Lobačevskiĭ. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 253–273 (1953). (Hungarian)

Varga, Ottó. The effect of the geometry of Bolyai-Lobačevskiĭ on the development of geometry. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 151–171 (1953). (Hungarian)

Kalmár, László. The influence of the geometry of Bolyai-Lobačevskiĭ on the development of axiomatic method. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 235–242 (1953). (Hungarian)

Lecture given 18 Dec. 1952 at a celebration of the 150th anniversary of János Bolyai's birth.

***Cartan, Elie. Œuvres complètes. Partie II. Vol. 1. Algèbre, formes différentielles, systèmes différentiels. Vol. 2. Groupes infinis, systèmes différentiels, théories d'équivalence.** Gauthier-Villars, Paris, 1953. Vol. 1: ix+pp. 1–561 (1 plate); vol. 2: v+pp. 563–1384. Bound 5500 francs; unbound 4800 francs.

HAVE

See the review of Partie I [these Rev. 14, 343] for a description of the general arrangement of papers.

***Dini, Ulisse. Opere. Vol. I. Algebra. Geometria differenziale.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese della Casa Editrice Perrella, Roma, 1953. 698 pp. (1 plate). 6000 lire.

HAVE

This is the first of a projected three-volume edition of Dini's mathematical papers. Misprints in the original papers have been corrected; corrections and comments on the papers are given in footnotes by the editors. The present volume contains a commemorative essay by L. Bianchi [Ann. Univ. Toscane (N.S.) 7(41), 155–169 (1922)], a list of Dini's publications, his works on algebra and differ-

ential geometry and essays by M. Cipolla on the algebraic papers and by Enea Bortolotti on the differential geometric papers.

Félix, Lucienne. Quelques aspects de l'histoire des mathématiques d'après les Leçons sur les Constructions Géométriques d'Henri Lebesgue. Rev. Gén. Sci. Pures Appl. 60, 265–276 (1953).

Kárteszi, Ferenc. Lobačevskiĭ's life and work. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 189–197 (1953). (Hungarian)

Rostand, François. Schopenhauer et les démonstrations mathématiques. Rev. Hist. Sci. Appl. **6**, 203–230 (1953).

Zorat, Alfredo. Decimazioni zoratiane. Giorn. Mat. Battaglini (5) **1**(**81**), 219–223 (1953).

FOUNDATIONS

***Dopp, Joseph. Leçons de logique formelle. Première partie. Logique ancienne. La logique des jugements prédicatifs.** Editions de l'Institut Supérieur de Philosophie, Louvain, 1949. xii+166 pp.

***Dopp, Joseph. Leçons de logique formelle. Deuxième partie. Logique moderne. I. Le calcul des propositions inanalysées.** Editions de l'Institut Supérieur de Philosophie, Louvain, 1950. xii+216 pp.

***Dopp, Joseph. Leçons de logique formelle. Troisième partie. Logique moderne. II. Logique des propositions à une ou plusieurs mentions d'objets.** Editions de l'Institut Supérieur de Philosophie, Louvain, 1950. xvi+274+15 pp.

This text-book is intended primarily for students in the humanities rather than for students in the sciences or mathematics. It contains an account of the classical syllogistic logic (Part I), of the modern two-valued propositional calculus (Part II), and of the functional calculus of first order (Part III). The propositional calculus is developed first by matrix methods and then axiomatically. Both pure and applied forms of the functional calculus are considered. This book should prove valuable for logic courses in philosophy departments in French-speaking universities.

R. M. Martin (Philadelphia, Pa.).

HAVE

***Tarski, Alfred. Undecidable theories. In collaboration with Andrzej Mostowski and Raphael M. Robinson.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, Amsterdam, 1953. xi+98 pp. 9 florins.

The book gives an introductory account of the methods introduced by Tarski for establishing the undecidability of several fairly simple branches of mathematics (group theory, lattices, abstract projective geometry, closure algebras and others). The methods and aims of this work are probably more easily intelligible and more interesting to the 'ordinary' mathematician than those of any other branch of mathematical logic.

Notation. The systems (here called 'theories') are all formalized in the predicate calculus of first order with identity; the set of formulae is (general) recursive; if the set of axioms is recursive too, the theory is axiomatizable; a theory T_2 is an extension of the (sub) theory T_1 if every valid formula of T_1 is valid in T_2; a theory is essentially undecidable if every axiomatizable consistent extension is undecidable. A constant of T_2, e.g., a function symbol f, is (formally) definable in T_1 with respect to (a common extension) T if there is an expression Φ of T_1 such that $(\wedge x)[f(x)=\Phi(x)]$ is valid in T (p. 20). The (metamathematical) definability of a function $F(n)$ is introduced on p. 40: a sequence of symbols $\Delta_0, \Delta_1, \cdots$ of T_1 and an expression Φ such that for each integer n, $(\wedge v)[\Phi(\Delta_n, v)\leftrightarrow v=\Delta_{F(n)}]$ is valid in T_1; all functions which are metamathematically definable in an axiomatizable system, are general recursive. $T^{(P)}$ is the theory T in which the variables are required to satisfy the condition P. A theory T_2 is (weakly) interpretable in T_1 if the axioms of T_2 are provable from (consistent with) the axioms of T_1 and suitable definitions of the constants of T_2 in T_1; T_2 is relatively interpretable in T_1 if $T_2^{(P)}$ is interpretable in T_1.

In part I simple general results are obtained, e.g., if T_2 is a finite extension of T_1 and is undecidable, so is T_1 (Theorem 5), or: if T_2 is essentially undecidable, finitely axiomatizable and weakly interpretable in T_1 then every subtheory T of T_1 is undecidable (Theorem 8), T_1, T_2 and T all having the same constants.

For the application of these theorems it is necessary to have simple theories which are known to be essentially undecidable: One, due to R. M. Robinson [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 729–730], is finitely axiomatizable: $Sx=Sy\rightarrow x=y$, $0\neq Sy$, $x\neq 0\rightarrow Vy(x=Sy)$, $x+0=x$, $x+Sy=S(x+y)$, $x\cdot 0=0$, $x\cdot Sy=(x\cdot y)+x$; an even weaker one (R), p. 53, has only a schema for numerical axioms. Both these systems for the theory of natural numbers are essentially undecidable since all recursive functions are (metamathematically) definable in them. The (elementary) theory of arbitrary integers is essentially undecidable since the theory of natural numbers can be relatively interpreted in the theory of arbitrary integers. A slightly longer argument establishes the essential undecidability of the theory of non-densely ordered rings.

In part III group theory

$$G\,[x\circ(y\circ z)=(x\circ y)\circ z,\ Vz(x=y\circ z),\ Vy(x=y\circ z)]$$

is shown to be undecidable (it is not essentially undecidable since the theory of commutative groups is decidable); the theory of arbitrary integers is relatively interpretable in an extension G_1 of G; the consistency of G_1 is established by the following model: the variables x, y, z range over biunique functions of the integers, $(x\circ y)(n)=x[y(n)]$, the 'integers' in the model are those functions which commute with the successor function, and y 'divides' x if all z which commute with x also commute with y.

The book puts more emphasis on simple general results than on the much more difficult constructions which are needed before these results can be applied: thus it fails to give a full picture of the impressive work done by the authors and their collaborators, but, on the other hand, is suitable for a wide class of readers.

[Reviewer's note. In connection with the improvement of Theorems 5 and 8 Tarski asks (p. 19) (i) whether Theorem 5 is true for arbitrary axiomatizable extensions, (ii) whether every essentially undecidable axiomatizable theory has a finitely axiomatizable subtheory which is essentially undecidable. The answers are negative: (i) the monadic

predicate calculus M with the constants A_i, c_i, $i=0, 1, \cdots$ is decidable; if $P_i(n)$ is a recursive enumeration of primitive recursive predicates, the following axioms are recursive: $A_i(c_j)(\sim A_i(c_j))$ if $P_i(j)$ holds (does not hold); M extended by these axioms is (essentially) undecidable. Tarski's problem remains open for theories with a finite number of constants. (ii) The system (R) mentioned above, has no essentially undecidable subtheory since every such subtheory has a finite model.] *G. Kreisel* (Reading).

Dekker, J. C. E. Two notes on recursively enumerable sets. Proc. Amer. Math. Soc. **4**, 495–501 (1953).

The first of these two notes is concerned with establishing some algebraic properties of simple and hypersimple sets. More particularly, it is shown that the product of two simple (hypersimple) sets is simple (hypersimple). Likewise the sum is simple (hypersimple) or has a finite complement. There exist two simple (hypersimple) sets whose sum is the set of non-negative integers. And the sum of the class of simple (hypersimple) sets with the class of sets whose complements are finite is a dual ideal in the class of all recursively enumerable sets.

The second note defines notions of productive and creative sets. A set a is then mesoic if and only if it is recursively enumerable but not recursive, not simple, and not creative. The existence of a mesoic set is established.

R. M. Martin (Philadelphia, Pa.).

Rasiowa, H. Algebraic treatment of the functional calculi of Heyting and Lewis. Fund. Math. **38**, 99–126 (1951).

Every formula α of a functional calculus can be interpreted as a functional Φ_α on an abstract set I with values in a suitable abstract algebra $\mathfrak{A}$. Every formula of the ordinary first order functional calculus can be interpreted as such a functional with $\mathfrak{A}$ a complete Boolean algebra, every formula of Heyting's functional calculus [S.-B. Preuss. Akad. Wiss. **1930**, 42–56] if $\mathfrak{A}$ is a complete Brouwerian algebra, every formula of Lewis' $S4$ [Lewis and Langford, Symbolic logic, Century, New York-London, 1932] if $\mathfrak{A}$ is a complete closure algebra. By methods similar to those of Rasiowa and Sikorski [Fund. Math. **37**, 193–200 (1950); these Rev. **12**, 661], the author then proves the following theorems which are generalizations of theorems by Henkin, MacKinsey and Tarski: (I) Let I_0 be the set of all positive integers. There exists a complete Brouwerian algebra $\mathfrak{B}_0$ such that a formula α of Heyting's functional calculus is provable if and only if the corresponding functional Φ_α is identically equal to the zero-element of $\mathfrak{B}_0$, and there is a complete closure algebra $\mathfrak{C}_0$ such that a formula α of Lewis's calculus is provable if and only if the corresponding functional Φ_α is equal to the unit element of $\mathfrak{C}_0$. (II) A formula α of the functional calculus of Heyting is provable if and only if, for every non-empty set I and every complete Brouwerian algebra $\mathfrak{B}$, the functional Φ_α is identically equal to the zero element of $\mathfrak{B}$; a formula α of the functional calculus of Lewis is provable if and only if, for every non-empty set I and every complete closure algebra $\mathfrak{C}$, the functional Φ_α is identically equal to the unit element of $\mathfrak{C}$. *I. Novak Gál* (Ithaca, N. Y.).

Marcus, Ruth Barcan. Strict implication, deducibility and the deduction theorem. J. Symbolic Logic **18**, 234–236 (1953).

The following form of the deduction theorem is proved for $S4$ and $S5$ of Lewis: If $A \vdash B$, then $\vdash A \prec B$; if $A_1, \cdots, A_n \vdash B$, then $\vdash (A_1 \& \cdots \& A_n) \prec B$. In an earlier paper [same J. **11**, 115–118 (1946); these Rev. **8**, 306] the author proved: If $A_1, \cdots, A_{n-1}, A_n \vdash B$, then $A_1, \cdots, A_{n-1} \vdash A_n \prec B$, under a restriction concerning the A's; from matrix III on p. 493 of Lewis and Langford, "Symbolic logic" [Century, New York-London, 1932], it follows that this theorem for $n>1$ is not true without the restriction. *A. Heyting.*

Church, Alonzo. Non-normal truth-tables for the propositional calculus. Bol. Soc. Mat. Mexicana **10**, 41–52 (1953). no. 1-2

The author first points out that any system of truth-tables having the two-element Boolean algebra as a homomorphic image will be a characteristic system of the two-valued propositional calculus if the designated truth-values are those which have the unit of the Boolean algebra as their image. He calls characteristic systems of this kind normal in the sense of Carnap and all others weakly non-normal. He then points out that a weakly non-normal characteristic system may be obtained from an arbitrary Boolean algebra by taking the unit as the single designated truth-value and shows that every characteristic system is either a Boolean algebra or reduces to one under a homomorphism if we assume that the truth-value of $p \supset q$ is undesignated when those of p, q are designated, undesignated respectively, i.e. that the truth-table of $p \supset q$ is regular.

The author calls a characteristic system for which the regularity condition fails strongly non-normal and gives an example of such a system together with an example due to Z. P. Dienes [J. Symbolic Logic **14**, 95–97 (1949); these Rev. **11**, 1]. He raises the questions of the existence of other strongly non-normal systems and of a survey of them but does not discuss them in detail. *A. Rose.*

Beth, E. W. On Padoa's method in the theory of definition. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 330–339 (1953).

Let K be a set of propositions in the restricted predicate calculus which contains a relation of n variables $a(x_1, \cdots, x_n)$ and other relations $t_1, t_2, \cdots$. Then a is said to be definable with respect to K in terms of $t_1, t_2, \cdots$ if there exists a predicate $U(x_1, \cdots, x_n)$ (i.e., a well-formed formula of n free variables) which contains only the relations $t_1, t_2, \cdots$ such that the proposition "for all $x_1, \cdots, x_n$, a is equivalent to U" is derivable from K. Let $b(x_1, \cdots, x_n)$ be a relation which is not contained in K and let K^* be the set obtained from K by replacing a everywhere by b. Finally, let X be the proposition "for all $x_1, \cdots, x_n$, a is equivalent to b". Then the important central result of the paper under review is that a is definable with respect to K in terms of $t_1, t_2, \cdots$ if and only if X is derivable from the union of K and K^*. For the proof, use is made of the general principle that the deducibility of X from the union of K and K^* implies that X is deducible also from the conjunction of a finite number of propositions of K and K^*. A careful analysis of the derivation after the manner of Gentzen's theory of demonstration then leads to the desired result.

A. Robinson (Toronto, Ont.).

Rasiowa, H., and Sikorski, R. A proof of the Skolem-Löwenheim theorem. Fund. Math. **38**, 230–232 (1951).

In this paper the authors adapt the methods of an earlier paper [Fund. Math. **37**, 193–200 (1950); these Rev. **12**, 661] to give a simple proof of the Skolem-Löwenheim theorem by topological methods: Every consistent set A of formulae of the classical (first order) functional calculus

is simultaneously satisfiable in the domain I of positive integers. *I. Novak Gál* (Ithaca, N. Y.).

Härtig, Klaus. Über die Struktur der klassischen Syllogistik. Wissensch. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **2**, 165–189 (1953).

This is a study of the traditional theory of the categorical syllogism regarded as an abstract system. The author adopts postulates and definitions motivated by the traditional theory, and then studies the resulting system by mathematical methods. The investigation is thus of similar character to those of J. Łukasiewicz [Aristotle's syllogistic from the standpoint of modern formal logic, Oxford, 1951; these Rev. **14**, 713], I. M. Bochenski [Dominican Studies **1**, 35–57 (1948)] and Słupecki [cited by Łukasiewicz]. The authors work is, however, stated to be independent of this Polish work; and there are some differences of detail which make the work of some consequence for those with a special interest in the subject. *H. B. Curry.*

Vaccarino, Giuseppe. Le logiche polivalenti e non aristoteliche. Archimede **5**, 226–231 (1953).

Feys, Robert. A simplified proof of the reduction of all modalities to 42 in S3. Bol. Soc. Mat. Mexicana **10**, 53–57 (1953).

The author denotes possibility and negation by M, N respectively and first shows that any modality can be expressed by a sequence of symbols being (at most) alternatively N, and M or MM. He calls such modalities simplified modalities. He then divides simplified proper modalities into four types and shows that those of the first type, i.e. those beginning with N and ending with M, can be reduced to ten. It follows from this that those of each of the other three types can be reduced to ten. The forty proper modalities plus the two improper modalities, p and Np, give the 42 modalities. *A. Rose* (Nottingham).

***Quine, W. V. Three grades of modal involvement.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 65–81. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author makes a detailed philosophical examination of the modal operator necessity. Other modal operators are not, for the most part, discussed, since they can be defined in terms of necessity. Necessity is considered as a semantical predicate attributable to statements as notational forms, as a statement operator which attaches to statements themselves and as a sentence operator which may be attached to open sentences, such as '$x>5$', preparatory to the ultimate attachment of quantifiers. *A. Rose* (Nottingham).

***Behmann, Heinrich. Die typenfreie Logik und die Modalität.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 88–96. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

The author raises objections to the modal logics of Lewis and to the three-valued modal logic of Łukasiewicz. He develops a modal logic whose interpretation is based on the idea of "all possible worlds" and discusses the need for the use of the truth-value "meaningless" in addition to the two usual ones. He goes on to discuss the view that there are exactly twelve distinct modalities. *A. Rose.*

Schütte, Kurt. Zur Widerspruchsfreiheit einer typenfreien Logik. Math. Ann. **125**, 394–400 (1953).

This paper reformulates a type-free system of logic [due to W. Ackermann, Math. Z. **55**, 364–384 (1952); these Rev. **14**, 344] in which set-theory can be developed and gives a new proof of its consistency. The law of the excluded middle does not hold in this system although many special cases of it do hold. *I. Novak Gál* (Ithaca, N. Y.).

Bar-Hillel, Yehoshua, and Carnap, Rudolf. Semantic information. British J. Philos. Sci. **4**, 147–157 (1953).

In a fixed language system L_n^π with n primitive individuals and π primitive one-place functions one considers the weakest propositions (disjunctions of positions and negations of the πn possible primitive propositions). These weakest propositions are called "content-elements". The class of content-elements implied by a proposition is the "content" of the proposition. In a natural way measure functions in the system of all contents are introduced and notions of information theory are applied. *H. Freudenthal* (Utrecht).

***von Wright, Georg Henrik. A treatise on induction and probability.** Harcourt, Brace and Co., Inc., New York, N. Y., Routledge and Kegan Paul Ltd., London, 1951. 310 pp. $5.00.

A philosophical book on induction and probability. *J. Łos* (Toruń).

***Juhos, Béla. Wahrscheinlichkeitsschlüsse als syntaktische Schlussformen.** Actes du XIème Congrès International de Philosophie, Bruxelles, 20–26 Août 1953, vol. XIV, pp. 105–108. North-Holland Publishing Co., Amsterdam; Editions E. Nauwelaerts, Louvain, 1953.

Some remarks on Carnap's book "The continuum of inductive methods" [Univ. of Chicago Press, 1952; these Rev. **14**, 4]. According to the author, the choice of a probability function is never performable by means of logical criteria only, but always some inductive principles must be used. Therefore the "Logik der Wahrscheinlichkeitsschlüsse" cannot be termed "Induktionslogik". *J. Łos* (Toruń).

Kemeny, John G. A logical measure function. J. Symbolic Logic **18** (1953), 289–308 (1954).

Laplace's definition of probability (favourable/possible) is extended in two directions: A proposition W is embedded in a formal system (S), M_n is the number of non-isomorphic models of (S) with exactly n individuals, M_n^W is the number of such models in which W is valid. (S) is required to have a finite number of constants, and to possess a finite model. The sequence $\{M_n^W/M_n\}$, denoted by $m(W)$, is (called) a measure of W, and $m(W)\gtreqless m(W')$ if for almost all n, $M_n^W\gtreqless M_n^{W'}$. If one of these relations holds, W and W' are comparable.

In previous measures the limit of the sequence was used instead of the sequence itself: the latter is a refinement of the former, analogous to a non-Archimedean number system. The usual axioms of probability apply to the measure (p. 298). No general results on the asymptotic behaviour of $m(W)$ are established except for the monadic predicate calculus: all W are effectively comparable; this is not true for the predicate calculus with a (single) relation. The author explains how such notions as 'degree of factual support' or 'degree of confirmation' may be defined in terms of his measure.

The author's definition refers to his "Models of logical systems" [same J. **13**, 16–30 (1948); these Rev. **9**, 487], but the applications use only predicate logics where his definition coincides with the familiar one. The practical scope of the measure is not quite clear, since the author's hints are a little unrealistic: 'consider the information contained in a scientific text book' . . . 'in a long W' (p. 299).

G. Kreisel (Reading).

Williams, Donald C. On the direct probability of inductions. Mind **62**, 465–483 (1953).

In an earlier publication [The ground of induction, Harvard Univ. Press, 1947, pp. 98–99] the author presents the following controversial argument: "Since we know a priori that the great majority of sets [or possible samples] which are choosable from a population are statistically similar to that population, it is highly probable that the group which we actually draw as a sample is statistically similar to it, and hence that the population is statistically similar to the sample . . .". W. C. Kneale criticised the argument in a review [Philosophy **24**, 86–88 (1949)], saying that the author "covertly assumed an oversimplified principle for the inversion of probability statements". The present paper is mainly an answer to Kneale's criticism.

L. J. Savage (Chicago, Ill.).

Wilder, R. L. The origin and growth of mathematical concepts. Bull. Amer. Math. Soc. **59**, 423–448 (1953).

The development of mathematical ideas, especially that of the topological concept of curve, is treated from the individual and group aspect. *H. Freudenthal* (Utrecht).

★Duhem, Pierre. The aim and structure of physical theory. Foreword by Louis de Broglie. Translated by Philip P. Wiener. Princeton University Press, Princeton, N. J., 1954. xxii+344 pp. $6.00.

Translated from "La théorie physique, son objet, sa structure" [2d ed., M. Rivière et Cie, Paris, 1915].

ALGEBRA

Andersen, Erik Sparre. Two summation formulae for product sums of binomial coefficients. Math. Scand. **1**, 261–262 (1953).

$$\sum_{i=0}^{k}\binom{a}{i}\binom{-a}{n-i}=\frac{n-k}{n}\binom{a-1}{k}\binom{-a}{n-k}\quad\text{for } n\geqq 1,\ 0\leqq k\leqq n;$$

and a similar one for $\sum_{i=0}^{k}\binom{a}{i}\binom{1-a}{n-i}$. The proof is by induction on k. Used in another paper [Math. Scand. **1**, 263–285 (1953); these Rev. **15**, 444]. *K. L. Chung.*

Erdös, Paul. The number of multinomial coefficients. Amer. Math. Monthly **61**, 37–39 (1954).

It is proved that the number of multinomial coefficients

$$\frac{n!}{i_1!\cdots i_r!(n-k)!}\quad\left(\sum_{j=1}^{r} i_j=k;\ i_1\leqq\cdots\leqq i_r\leqq n-k\leqq n-2\right)$$

less than x is $(1+2^{1/2})x^{1/2}+o(x^{1/2})$. It is remarked that a more detailed analysis would yield the estimate

$$(1+2^{1/2})x^{1/2}+c_3x^{1/3}+\cdots+c_mx^{1/m}+o(x^{1/m}).$$

L. Carlitz (Durham, N. C.).

Touchard, Jacques. Permutations discordant with two given permutations. Scripta Math. **19**, 109–119 (1953).

Exposé abrégé d'un mémoire de l'auteur, contenant les démonstrations des résultats publiés par lui il y a vingt ans [C. R. Acad. Sci. Paris **198**, 631–633 (1934)] et qui furent le point de départ d'une série de recherches originales depuis Erdös et Kaplansky [Amer. J. Math. **68**, 230–236 (1946); ces Rev. **7**, 407] jusqu'à K. Yamamoto [Sûgaku **2**, 159–162 (1949); Jap. J. Math. **21**, 113–119 (1952); ces Rev. **14**, 442]. Etant données deux permutations quelconques A et B des éléments $1, 2, \cdots, n$, on cherche le nombre des permutations, C, telles que, a, b, c désignant les éléments de même rang dans A, B, C respectivement, c soit toujours distinct de a et de b ($c\neq a, b$). Une solution formelle, au moyen des fonctions génératrices, est déduite du lemme: Le nombre des manières de choisir k éléments parmi les éléments $1, 2, \cdots, n$, disposés circulairement, de façon que deux quelconques d'entre eux ne soient pas consécutifs, est: $n\binom{n-k}{k}\Big/(n-k)$. La démonstration reproduite par le rédacteur de la note est celle de Kaplansky [Bull. Amer. Math. Soc. **49**, 784–785 (1943); ces Rev. **5**, 86], mais il est peu vraisemblable que l'auteur n'ait pas eu, dès 1934, la preuve d'une proposition aussi simple. Soit S la substitution telle que $A\cdot S=B$. Si S est $x\to 1+x$ (mod. n) une solution explicite est donnée [cf. Schöbe, Math. Z. **48**, 781–784 (1943); ces Rev. **5**, 29]. Le cas où S est $x\to 1-x$ (mod. n) et celui où S est le produit de deux substitutions circulaires sont examinés, la méthode fournissant, pour le premier, une expression plus simple que le résultat connu. Etude de la fonction génératrice dans le cas général.

A. Sade (Marseille).

Mendelsohn, N. S. A problem in combinatorial analysis. Trans. Roy. Soc. Canada. Sect. III. **47**, 21–26 (1953).

Le problème des combinaisons avec répétitions restreintes semble avoir été posé pour la première fois par P. R. de Montmort [Essay d'analyse sur les jeux de hazard, 2ᵉ ed., Quillau, Paris, 1713, p. 56]. Sergescu [Acad. Roum. Bull. Sect. Sci. **23**, 485–491 (1941); ces Rev. **3**, 259], puis Montel [C. R. Acad. Sci. Paris **214**, 139–141 (1942); Bull. Sci. Math. (2) **66**, 86–103 (1942); ces Rev. **4**, 184; **6**, 88] en ont donné une solution explicite. Il se formule ainsi: Quel est le nombre $C_r(a, b, c, \cdots, m)$ des combinaisons r à r des éléments $A, B, C, \cdots, M$, où A figure au plus a fois, B au plus b fois, $\ldots$, M au plus m fois? Ici on examine le cas où les nombres $a, b, c, \cdots, m$ ne sont pas tous inégaux. L'équation aux différences de $C_r(a, b, c, \cdots, m)$ est établie et la solution explicite quand l'ensemble $a, b, c, \cdots, m$ se compose de i termes égaux à a et de j termes égaux à b est donnée, puis justifiée. Cas où tous les nombres $a, b, c, \cdots, m$ sauf deux, surpassent $r-1$; exemples; problème analogue pour les permutations. Les exemples proposés montrent que le calcul numérique de C_r réalise une économie par rapport aux méthodes anciennes.

A. Sade (Marseille).

Moser, Leo. Note on a combinatorial formula of Mendelsohn. Trans. Roy. Soc. Canada. Sect. III. **47**, 27 (1953).

Démonstration, au moyen de la fonction génératrice, de l'expression de $C_r(a^i, b^j)$ signalée dans l'analyse précédente.

A. Sade (Marseille).

Mohr, Ernst. Einfacher Beweis des verallgemeinerten Determinantensatzes von Sylvester nebst einer Verschärfung. Math. Nachr. **10**, 257–260 (1953).

The theorem of the title is as follows. Form all the $m\times m$ matrices which contain as minors a fixed principal $h\times h$ matrix A_h of the $n\times n$ matrix A; and form the compound matrix S which has as elements the determinants of these $\binom{n-h}{m-h}\times\binom{n-h}{m-h}$ m-rowed minors. The value of det S is the product of the $\binom{n-h-1}{m-h}$ power of det A_h by the $\binom{n-h-1}{m-h-1}$ power of det A. The rank of S is 0 unless the set of h row [column] vectors of A on the vectors corresponding to the h rows being used above is maximal; in the latter case, the rank of S is $\binom{a-2h+b}{m-2h+b}$, where a is rank A, b is rank A_h. The proof uses the principle of irrelevance of algebraic identities; and further considers the matrix obtained from A by reducing $n-1$ elements of one column to 0 by elementary operations, a powerful general method for proving identities concerning determinants.

J. L. Brenner (Pullman, Wash.).

Stojaković, Mirko. Quelques applications des déterminants rectangulaires aux produits intérieurs et extérieurs des matrices. C. R. Acad. Sci. Paris **237**, 688–690 (1953).

In two prior articles [Bull. Soc. Math. Phys. Serbie **4**, nos. 1–2, 9–23 (1952); C. R. Acad. Sci. Paris **236**, 877–879 (1953); these Rev. **14**, 443, 1055], the author defined determinants of rectangular matrices. He now applies these to inner and outer products of rectangular matrices. If A and B are matrices of orders $l_1\times k$ and $l_2\times k$, the author defines the inner product $\langle A, B\rangle$ to be det (AB'), and the norm $N(A)$ to be $\langle A, A\rangle$. He defines the outer product of vectors $\mathbf{a}_1\times\mathbf{a}_2\times\cdots\times\mathbf{a}_l$ $(l\leq k-1)$ in k-space to be a vector $\mathbf{b}_0$, each element of which is a certain determinant. Several properties of the products are announced.

G. E. Forsythe (Los Angeles, Calif.).

Wegner, U. Bemerkungen zur Matrizentheorie. Z. Angew. Math. Mech. **33**, 262–264 (1953).

This is an abstract of a lecture of partly expository character intended to give a uniform representation of certain matrix operations used in applied problems. Particular attention is given to the usefulness of the Horner polynomials $h_i(x)$ and the analogous matrix polynomials $H_i(A)$ defined recursively by $h_{i+1}(x)=xh_i(x)-(-1)^iC_{i+1}$ where $x^n-c_1x^{n-1}+\cdots\pm c_n$ is the characteristic polynomial of the $n\times n$ matrix A considered. The H_i can be obtained without the knowledge of the c_i, in fact,

$$ic_i=(-1)^{i-1}\text{ trace }(AH_{i-1}).$$

[See also Wegner, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 1933 and C. R. Acad. Sci. Paris **228**, 1200 (1949); these Rev. **10**, 586.]

O. Taussky-Todd (Washington, D. C.).

Egerváry, Jenő. Canonical representation of matric functions and several applications. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 417–458 (1953). (Hungarian)

The paper is mainly expository. The author emphasizes the applicability of matrix theory; he gives several examples, some numerical and some theoretical (e.g., the canonical representation of cyclic matrices is applied to prove the ergodicity of certain Markoff chains). Much of the exposition is in terms of dyads; thus, for instance, the rank of a matrix A is defined as the minimum number of dyads whose sum is A. Canonical forms are derived not for individual matrices but for matrix functions (power series).

P. R. Halmos (Chicago, Ill.).

Ōtsuki, Tominosuke. Some theorems on a system of matrices and a geometrical application. Math. J. Okayama Univ. **3**, 89–94 (1953).

The space $\mathfrak{M}_{n,m}$ of all (n, m)-matrices can be bilinearly paired into that of all (n, n)-matrices by the multiplication $M_1\circ M_2=M_1M_2'$, where M_2' denotes the transpose of M_2. A linear subspace of $\mathfrak{M}_{n,m}$ is called an α-system, if the product of any two of its elements is commutative under this multiplication. To any linear subspace $\mathfrak{M}$ of $\mathfrak{M}_{n,m}$ the author defines an α-system $\alpha(\mathfrak{M})$ and an exterior quadratic form $\varphi(\mathfrak{M})$ associated with it. It is proved that: (1) If $\mathfrak{N}$ is an α-system in $\mathfrak{M}_{n,m}$, then dim $\mathfrak{N}\leq m$. (2) If $\mathfrak{M}$ is an r-dimensional linear subspace in $\mathfrak{M}_{n,m}$ and k the minimum number of variables in terms of which $\varphi(\mathfrak{M})$ can be expressed, then dim $\alpha(\mathfrak{M})=r-k\leq m$. The second theorem is applied to prove a result of the reviewer and N. H. Kuiper on the isometric imbedding of a compact Riemann manifold in an Euclidean space [Ann. of Math. (2) **56**, 422–430 (1952); these Rev. **14**, 408].

S. Chern.

Meserve, Bruce E. Irriducibilità del risultante e del discriminante. Boll. Un. Mat. Ital. (3) **8**, 243–252 (1953).

Let

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n\quad(n>0),$$
$$g(x)=b_0x^m+b_1x^{m-1}+\cdots+b_m\quad(m>0)$$

be two polynomials whose coefficients are indeterminates over a field K. When regarded as polynomials in the a_i and b_j the resultant $R(f, g)$ and the discriminant $D(f)$ have rational integral coefficients, and it is known that R and D are irreducible when K is of characteristic zero. The author considers the question of reducibility when K is replaced by the residue ring I_k, where k is a positive integer. The presence of divisors of zero makes it necessary to define reducibility so that $F=PQ$ is reducible only if deg $P>0$, deg $Q>0$ and deg $F=$ deg $P+$ deg Q. (In a footnote the editors of the journal point out that one further, rather elaborate, condition should be added without which one of the author's arguments appears to be doubtful.)

The purpose of the article is to show that, with this modified meaning of reducibility, the resultant remains irreducible over I_k for any k, and that the discriminant is irreducible over I_k if $k\nmid 4$ but is a square of an irreducible polynomial if $k=2$ or 4.

W. Ledermann (Manchester).

11/24/53

***Libois, Paul. Contenu projectif des notions de nombre et de tenseur.** Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 673–678. Akadémiai Kiadó, Budapest, 1952. (Hungarian and Russian summaries)

The author first recalls his previous researches concerning the "projective content" (contenu projectif) of the notion of number (i.e., element of a field), which have led him to an axiomatic definition of the projective one-dimensional spaces [Assoc. Française Avancement Sci., 64me session, Paris, Congrès de la Victoire, 1945, t. II, pp. 77–82]. He now deals with a question which derives from the same

general idea, although it is quite different technically: the question of the projective content of the notion of tensor. A tensor of a given kind is, according to the usual definition, a point of a vector space V; the projective content of that tensor will be a point of a projective space, namely the space at infinity of V. One may study that way, from a projective viewpoint, all problems regarding tensors, or finite-dimensional representations of groups [cf. also H. Weyl, The classical groups, Princeton, 1939, pp. 112–114; these Rev. 1, 42]. As an example of this, the author briefly exposes the projective content of the theory of finite-dimensional irreducible representations of the 3-dimensional rotation group, esp. what concerns the reduction of the Kronecker product of two such representations [cf. E. Cartan, Leçons sur la théorie des spineurs, t. I, II, Hermann, Paris, 1938]; this leads to quite elegant geometrical developments involving normal curves and Segre varieties.

J. Tits (Brussels).

Aczél, J. Bemerkungen über die Multiplikation von Vektoren und Quaternionen. Acta Math. Acad. Sci. Hungar. 3 (1952), 309–316 (1953). (Russian summary)

Postulates are stated for the scalar, vector, and quaternion products of vectors. The strongest of these postulates appear to be the distributive laws. The classical descriptions follow and are arrived at in part by the solution of functional equations. *F. Kiokemeister.*

Abstract Algebra

Morinaga, Kakutaro, and Nishigōri, Noboru. On axiom of betweenness. J. Sci. Hiroshima Univ. Ser. A. 16, 177–221, 399–408 (1952).

M. Altwegg [Comment. Math. Helv. 24, 149–155 (1950); diese Rev. 12, 237] hat ein Axiomensystem der teilweisen Ordnung unter Verwendung der Relation "zwischen" formuliert. Damit ist das Problem 1 von Birkhoffs "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; diese Rev. 10, 673] gelöst worden. Ein anderes Axiomensystem dieser Art ist von M. Sholander [Proc. Amer. Math. Soc. 3, 369–381 (1952); diese Rev. 14, 9] gegeben worden.

Die Verfasser stellen einige Axiomensysteme für die Relation "zwischen" auf, ohne die Arbeit von Altwegg zu erwähnen. Ihr Axiomensystem $\mathfrak{B}$ ist dem System von Altwegg ähnlich, ist aber nicht unabhängig, da sich B7 aus den übrigen Axiomen ableiten lässt, sodass die Sätze 2 und 4 des Kapitels I unrichtig sind. Ebenso ist das Axiomensystem $\mathfrak{D}.\mathfrak{B}$ nicht unabhängig und der Satz des Kapitels III ist falsch. Da diese falschen Sätze weiter verwendet werden, dazu auch andere Unrichtigkeiten hinzutreten [z.B., wird im Beweise des Lemmas 4.3 des Kapitels II die Behauptung dieses Lemmas verwendet] und auch die Symbolik nicht immer klar ist, scheint es dem Referenten schwer zu entscheiden, inwieweit die Resultate richtig sind.

M. Novotný (Brno).

Rabin, Michael. A theorem on partially ordered sets. Riveon Lematematika 7, 26–29 (1954). (Hebrew. English summary)

The theorem of the title asserts that if M satisfies ascending and descending chain conditions and every totally unordered subset of M is finite, then M is finite. [Reviewer's note: The theorem is due to D. König. See Birkhoff, Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948, p. 39, ex. 7; these Rev. 10, 673.]

M. Jerison (Lafayette, Ind.).

Jónsson, Bjarni. On the representation of lattices. Math. Scand. 1, 193–206 (1953).

A representation (F, U) of a lattice A is an isomorphic mapping F of A onto a sublattice of the lattice of all equivalence relations over U. It is said to be of Type 1 if $x,y \in A$ implies $F(x)+F(y)=F(x);F(y)$ where ";" denotes the product of the relations $F(x)$ and $F(y)$, and "+" denotes the least equivalence relation containing both. It is of Type 2 if $F(x)+F(y)=F(x);F(y);F(x)$, and of Type 3 if $F(x)+F(y)=F(x);F(y);F(x);F(y)$. Thus Type 1 comprises the commuting equivalence relations. Theorems: If A has a representation of Type 2 then A is modular, and conversely. There exists a modular lattice (namely, all subgroups of a non-Desarguesian projective plane) without a representation of Type 1, and hence not isomorphic to a lattice of normal subgroups of a group. This is proved by showing an identity which must be satisfied by modular lattices of Type 1, answering problem 27 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; these Rev. 10, 673]. Every lattice has a representation of Type 3, strengthening a result of the reviewer [Bull. Amer. Math. Soc. 52, 507–522 (1946); these Rev. 8, 62]. *P. M. Whitman* (Silver Spring, Md.).

Raney, George N. A subdirect-union representation for completely distributive complete lattices. Proc. Amer. Math. Soc. 4, 518–522 (1953).

Definitions. 1. Let L be a partially ordered set. A subset M of L is called a semi-ideal of L if the relations $x \in M$, $y \leqq x$ imply $y \in M$. Let $R(L)$ be the complete lattice of semi-ideals of L. If L is a complete lattice, let, for $x \in L$, $K(x)$ denote the intersection of all semi-ideals $M \in R(L)$ such that $x \leqq \sup M$. 2. Let X be a set and σ a binary relation on X. Let $\sigma \circ \sigma$ denote the binary relation on X defined as follows: $x\sigma\circ\sigma y$ if and only if there exists a z such that $x\sigma z$ and $z\sigma y$. If $A \subset X$, let $\Phi(A)$ be the set of $x \in X$ such that there exists a $y \in A$ such that $x\sigma y$. Let $L(\sigma)$ be the family of all $\Phi(A)$ for $A \subset X$, partially ordered by set-inclusion. Theorems. 1. For a complete lattice L to be completely distributive it is necessary and sufficient that for every $x \in L$, $\sup K(x)=x$. 2. If σ is a binary relation on a set X and if $\sigma=\sigma\circ\sigma$, then $L(\sigma)$ is a completely distributive complete lattice. If, in addition, σ is reflexive, then $L(\sigma)$ is a complete ring of sets. 3. Every completely distributive complete lattice is isomorphic with a closed sublattice of the direct union of a family of complete chains. This result gives the answer to one of Birkhoff's questions [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 4–7; these Rev. 13, 718]. *O. Borůvka* and *M. Novotný* (Brno).

Enomoto, Shizu. Boolean lattices and set lattices. Sûgaku 5, 1–10 (1953). (Japanese)

An exposition announcing the result of the author's paper in Osaka Math. J. 5, 99–115 (1953); these Rev. 15, 108. By making use of the notion of the "ramification set" [Horn and Tarski, Trans. Amer. Math. Soc. 64, 467–497 (1948); these Rev. 10, 518], the author gives a detailed discussion of the representation of a Boolean algebra by an algebra of sets. In particular, theorem 3.1 proves the equivalence of the following four conditions for a Boolean algebra

A. (i) A is isomorphic to an algebra F of sets with the property: $S_0=\sum_{\mu\varepsilon\Delta}S_\mu$ if the smallest set S_0 containing S_μ, $\mu\varepsilon\Delta$, is in F. The isomorphism $s\to S$ is meant to be such that $\bigcup_{\mu\varepsilon\Delta}s_\mu\varepsilon A$ implies $\bigcup_{\mu\varepsilon\Delta}s_\mu\to\sum_{\mu\varepsilon\Delta}S_\mu$. (ii) For any $s\varepsilon A$, $s\neq0$, there exists a 0–1-valued finitely additive measure f with $f(s)=1$ such that $f(\bigcup_{\mu\varepsilon\Delta}s_\mu)=0$ if $f(s_\mu)=0$, $\mu\varepsilon\Delta$. (iii) A is atomic. (iv) A is completely distributive in a certain extended sense. This theorem gives an affirmative and an extended answer to the problem [Horn and Tarski, loc. cit.] of characterizing the atomicity of a Boolean algebra. Theorem 3.2 states that (i) and (ii) are still equivalent to each other even if we restrict the power $\bar{\bar{\Delta}}$ of Δ to be $\leqq\nu$. Moreover, if the power of every ramification set in A be $\leqq\nu$, the condition (i) or (ii) in the theorem 3.2 implies the atomicity of A. *K. Yosida* (Osaka).

Zacher, Giovanni. Sugli emimorfismi superiori ed inferiori tra reticoli. Rend. Accad. Sci. Fis. Mat. Napoli (4) **19** (1952), 45–56 (1953).

Detailed analysis along lines already announced by the author [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, vol. II, Perrella, Roma, 1953, pp. 251–252; these Rev. **15**, 4]. *P. M. Whitman* (Silver Spring, Md.).

Andreoli, Giulio. Funzioni simmetriche, involuzioni ed N-adi in un'algebra di Boole. Giorn. Mat. Battaglini (5) **1**(81), 181–198 (1953).

The author considers at length the interrelationships of symmetric functions of elements of a Boolean algebra. A special role is assigned to

$$P(a,b,c)=abc+ab'c'+a'bc'+a'b'c,$$
$$D(a,b,c)=a'bc+ab'c+abc'+a'b'c',$$

and the corresponding functions of other numbers of elements. Transformation and matrix notations are also used. *P. M. Whitman* (Silver Spring, Md.).

Preston, G. B. The arithmetic of a lattice of sub-algebras of a general algebra. J. London Math. Soc. **29**, 1–15 (1954).

A general algebra $[A,\Omega]$ is a set A of elements and a set Ω of operations on these elements, such that an operation f_n of Ω (of index n) assigns to every ordered set $a_1, a_2, \cdots, a_n$ of n elements of A a unique element $f_n(a_1, a_2, \cdots, a_n)$ of A. Different operations in Ω may have different indices. A subset of A is a subalgebra if it is closed under the operations of Ω. If $(i, j, \cdots, k)$ is a fixed subset of $(1, 2, \cdots, n)$, a subalgebra $\mathfrak{a}$ of A is an $(i, j, \cdots, k)$-ideal with respect to f_n if whenever $a_i, a_j, \cdots, a_k$ belong to $\mathfrak{a}$, $f_n(a_1, a_2, \cdots, a_n)\varepsilon\mathfrak{a}$. If r is a positive integer $\leqq n$, a subalgebra $\mathfrak{a}$ is an r-ideal if $f_n(a_1, a_2, \cdots, a_n)\varepsilon\mathfrak{a}$ when any r of $a_1, a_2, \cdots, a_n$ belong to $\mathfrak{a}$. Thus in an ordinary ring, if f_2 represents multiplication, the (1)-ideals, (2)-ideals, and 1-ideals with respect to f_2 are respectively the right ideals, left ideals and 2-sided ideals, while (1, 2) ideals and 2-ideals would be arbitrary subrings.

For fixed $i, j, \cdots, k$ the $(i, j, \cdots, k)$-ideals with respect to f_n form a complete lattice. So also do the r-ideals. An ideal $\mathfrak{a}$, of any type, is $(i, j, \cdots, k)$-prime with respect to f_n if $f_n(a_1, a_2, \cdots, a_n)\varepsilon\mathfrak{a}$ implies that $a_i, a_j, \cdots, a_k$ all belong to $\mathfrak{a}$. Similarly $\mathfrak{a}$ is r-prime if $f_n(a_1, a_2, \cdots, a_n)\varepsilon\mathfrak{a}$ implies that at least r of $a_1, a_2, \cdots, a_n$ belong to $\mathfrak{a}$. The 1-ideals with respect to f_n are investigated in detail and henceforth ideal means 1-ideal and prime means 1-prime. The radical of an ideal $\mathfrak{a}$ is defined and, under suitable assumptions of commutativity, associativity and distributivity, is shown to be the intersection of the minimal primes containing $\mathfrak{a}$. Primary ideals are defined and, again under suitable commutativity assumptions, the radical of a primary ideal is prime. The Noether uniqueness theorems for the representation of an ideal as an intersection of primary ideals are then proved.

Specializing to algebras with one binary operation, a complete Krull-Noether theory for ideals in commutative semigroups results. The general results also include the classical Krull-Noether theory for commutative rings (proved without using the group properties of addition), but not the corresponding results for non-commutative rings obtained by McCoy [Amer. J. Math. **71**, 823–833 (1949); these Rev. **11**, 311] and the reviewer [Canadian J. Math. **4**, 43–57 (1952); these Rev. **13**, 618]. *D. C. Murdoch* (Vancouver, B. C.).

Szász, Gábor. The independence of associative conditions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 569–577 (1953). (Hungarian)

Hungarian version of another paper [Acta Sci. Math. Szeged **15**, 20–28 (1953); these Rev. **15**, 95]. *P. R. Halmos* (Chicago, Ill.).

Rainich, G. Y. Ternary relations in geometry and algebra. Michigan Math. J. **1** (1952), 97–111 (1953).

Conditions nécessaires et suffisantes pour qu'une relation ternaire entre les variables x, y, z soit de la forme $x+y+z=0$, $+$ étant une loi de groupe abélien. Les considérations sur le rôle des relations ternaires en géométrie ne semblent pas bien originales [voir, en particulier, Huntington et Kline, Trans. Amer. Math. Soc. **18**, 301–325 (1917); Huntington, ibid. **26**, 257–282 (1924); Prenowitz, Ann. of Math. (2) **49**, 659–688 (1948); ces Rev. **10**, 57]. *J. Riguet* (Paris).

HAVE ***Northcott, D. G. Ideal theory.** Cambridge Tracts in Mathematics and Mathematical Physics, No. 42. Cambridge, at the University Press, 1953. viii+111 pp. $2.50.

This well-written book provides a self-contained treatment of certain portions of the modern theory of ideals in Noetherian rings, including the elements of the theory of local rings. No previous knowledge whatsoever of ring theory is assumed, and beginners to the subject will find here a very readable account. There is very little overlapping with McCoy's book, Rings and ideals [Open Court, Chicago, 1948; these Rev. **10**, 96], which is concerned mainly with rings not necessarily satisfying any chain condition. Indeed, to the reviewer's knowledge, there exists to date no other book which covers the same material ab initio.

After rings have been defined in a preliminary section, Chapter I begins with the stipulation that "ring" shall always mean commutative ring with identity. In this chapter are defined the elementary operations on ideals, the radical of an ideal, prime and primary ideals, isolated components of an ideal with respect to a multiplicative set. For ideals which are intersections of primary ideals the customary uniqueness theorems are proved and the representation of the radical is obtained. There follows a discussion of Noetherian rings, including the proof of the primary decomposition theorem and of the Hilbert basis theorem.

Chapter II is concerned with the standard properties of homomorphisms, residue class rings, and quotient rings, the last named including the generalized Chevalley-Uzkov quotient rings. Chapter III contains a further development of the ideal theory of Noetherian rings: theorems on the intersection of all the powers of an ideal, and of all the

symbolic powers of a prime ideal; proof of the uniqueness of the length of the composition series for a primary ideal; the principal ideal theorem; some theorems on the rank of an ideal.

The final two chapters deal mainly with local rings. The topics covered in the first half of Chapter IV include the dimension of a local ring, systems of parameters and their analytical independence, regular local rings. Integral dependence (for arbitrary rings) is now defined, and it is proved that every proper principal ideal in an integrally closed Noetherian integral domain is an intersection of symbolic powers of minimal prime ideals. With every minimal prime ideal there is associated a valuation of the quotient field of R, and some properties of these valuations are obtained. In point of fact, valuations are not mentioned by name at all, the author preferring to speak of "divisors" (formal integral linear combinations of minimal prime ideals). In addition, we note that in most of this theory of integrally closed rings the author has not eschewed zero divisors, but we have assumed R to be an integral domain for simplicity of statement.

Chapter V, "The analytic theory of local rings", deals, as its name implies, with those properties of a local ring in which convergence plays a role. The discussion here includes some properties of complete local rings, rings of formal power series, existence and uniqueness of the completion $\hat{Q}$ of a local ring Q, and some results on the relationship between the ideal theory in Q and $\hat{Q}$.

In connection with each chapter there is a supplementary note giving some historical remarks as well as some indication of further developments. There is also a bibliography at the end of the book.

The author has been hampered by lack of space, and some topics have had to be omitted entirely. It is rather unfortunate that one of these omissions is the dimension theory of polynomial rings, particularly since the hope is expressed in the preface that the book may be of assistance to those interested in modern algebraic geometry. The reading of current papers requires also some familiarity with a certain amount of technical apparatus, such as the language and elementary theory of modules, or of relatively prime ideals; neither topic is mentioned (except in the supplementary notes). It is, of course, true that inclusion of additional material of this sort might result, for reasons of space, in the displacement of some of the more interesting developments of ideal theory, but in a basic introductory book to a broad subject it is perhaps inevitable that a certain amount of moderately dull foundation underlie the more attractive superstructure. The fact remains, though, as indicated above, that the presentation is clear and makes pleasant reading, and that the book will encourage many who would not otherwise have done so to study ideal theory and algebraic geometry. *I. S. Cohen* (Cambridge, Mass.).

Northcott, D. G. On unmixed ideals in regular local rings. Proc. London Math. Soc. (3) **3**, 20–28 (1953).

Let R be a regular local ring, M its maximal ideal, A an ideal in R of rank r and generated by r elements. Generalizing a theorem of Macaulay for polynomial rings [Algebraic theory of modular systems, Cambridge, 1916], the reviewer has proved [Trans. Amer. Math. Soc. **59**, 54–106 (1946); these Rev. **7**, 509] that such an ideal A is unmixed, the proof making use of the structure theory of complete local rings. The author now gives a much more elementary treatment. For this purpose he makes use of a non-special basis of M, this being obtained by subjecting a given minimal basis of M to a linear transformation with indeterminate coefficients. It is then proved further that also every power of A is unmixed; this, too, had been done by Macaulay for polynomial rings. From this it is deduced that if P is a prime polynomial ideal, then any imbedded prime divisor of any power of P defines a singular subvariety of the variety of P. *I. S. Cohen* (Cambridge, Mass.).

Nagata, Masayoshi. Some remarks on local rings. Nagoya Math. J. **6**, 53–58 (1953).

L'auteur donne une démonstration correcte du résultat suivant [que le rapporteur avait incorrectement démontré; cf. P. Samuel, Ann. Inst. Fourier Grenoble **2**, 147–160 (1951); ces Rev. **13**, 579]: si I et J sont deux idéaux d'un anneau semi local A, et si $\hat{A}$ est le complété de A, alors $(I\cap J)\hat{A}=I\hat{A}\cap J\hat{A}$. Il démontre ensuite le théorème de normalisation par une méthode voisine de celle d'Uzkov [Mat. Sbornik N.S. **22**(**64**), 349–350 (1948); ces Rev. **9**, 562]: si y est un polynome non constant de $A=K[X_1, \cdots, X_n]$, il existe des polynomes $y_2, \cdots, y_n$ tel que A soit entier sur $K[y, y_2, \cdots, y_n]$ (K: corps infini ou non). Le théorème de Chevalley, "le complété d'un anneau local géométrique n'a pas d'éléments nilpotents", est généralisé à une classe d'anneaux locaux, contenant les anneaux locaux géométriques, et définie axiomatiquement. Enfin le théorème de normalité analytique de Zariski [Ann. Inst. Fourier Grenoble **2**, 161–164 (1951); ces Rev. **13**, 579] est étendu aux domaines locaux géométriques intégralement clos: le complété d'un tel domaine est un domaine intégralement clos. *P. Samuel* (Clermont-Ferrand).

Kerstan, Johannes. Bemerkungen zur Theorie der Quotientenringe. Wissensch. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **2**, no. 3–4, 9–11 (1953).

Let R be an arbitrary commutative ring with unit element. Suppose that $R_1, \cdots, R_n$ are a finite set of quotient rings of R [see Krull, Idealtheorie, Springer, Berlin, 1935] with the property that $R=R_1\cap R_2\cap \cdots \cap R_n$. Given an ideal $\mathfrak{a}$ in R its extension ideal in R_i is $\mathfrak{a}R_i$. The author establishes the following criterion: the ideal $\mathfrak{a}$ of R is exactly the intersection of its extension ideals $\mathfrak{a}R_i$, $i=1, \cdots, n$, when for every prime ideal $\mathfrak{p}$ belonging to $\mathfrak{a}$ at least one of the ideals $\mathfrak{p}R_i$ has the property that $R\cap \mathfrak{p}R_i=\mathfrak{p}$. As a corollary to this, if the ideal $\mathfrak{a}$ of R is the intersection of its extension ideals $\mathfrak{a}R_i$ then this is also true for the isolated component ideal of $\mathfrak{a}$. In case the R_i are all primary then every ideal of R is the intersection of its extension ideals in the R_i's. In particular, if the R_i are all primary and have the ascending chain condition, then the ascending chain condition prevails in R. The author also obtains some results about the relation of the isolated component ideals with certain sets of prime ideals. *I. N. Herstein.*

Johnson, R. E. The imbedding of a ring as an ideal in another ring. Duke Math. J. **20**, 569–573 (1953).

A ring R is said to be right faithful if $Ra=(0)$ with a in R only if $a=0$. Let $E(R)$ be the ring of all endomorphisms of the additive group of R, and let $N(R)$ be the largest subring of $E(R)$, containing R, in which R is a two-sided ideal. The author then establishes that if R, a right faithful ring, is contained as an ideal in a ring S then S is isomorphic to a subring of $N(R)\oplus S/R$ which contains R. This generalizes the well known result in case R has a unit element.

A right ideal I in a ring R is said to be weakly prime if whenever $aR\subset I$, $a\in R$, then $a\in I$. Let $A_r(R)$ be the set

of weakly prime right ideals of R (it is an "algebra" under infinite intersection). If R is an ideal of S let

$$A_r(S, R) = \{I \in A_r(S) \mid I \cap R \in A_r(R)\}.$$

The author proves that the mapping $I \to I \cap R$, $I \in A_r(S, R)$ is a homomorphism of the "algebra" $A_r(S, R)$ onto the "algebra" $A_r(R)$. Similar results are proved for various related notions of prime ideal. *I. N. Herstein.*

Herstein, I. N. The structure of a certain class of rings. Amer. J. Math. 75, 864–871 (1953).

The author's generalization [same J. 75, 105–111 (1953); these Rev. 14, 613] of a theorem of Jacobson, namely, that $a^{n(a)} - a$ in the center Z of a ring R for every $a \in R$ ($n(a) > 1$) guarantees that R is commutative, is extended as follows: if, for every $a \in R$, there exists a polynomial $p_a(t)$ such that $a^2 p_a(a) - a \in Z$, then R is commutative. For division rings a stronger result may be stated: R is commutative in case $a^{r+1} p_a(a) - a^r \in Z$ ($r > 0$ depending on a). The proof of this depends on a recent valuation-theoretic result of Nakayama [Canadian J. Math. 5, 242–244 (1953); these Rev. 14, 719]. As in the author's previous theorems of similar nature, the proof of the general case employs the result for division rings and the general structural theory of rings.

R. D. Schafer (Storrs, Conn.).

Kleinfeld, Erwin. Simple alternative rings. Ann. of Math. (2) 58, 544–547 (1953).

As a new identity for alternative rings, it is shown without any further restriction that fourth powers of commutators $xy - yx$ are in the nucleus. If a simple alternative ring is defined as one in which there is no proper two-sided ideal and which is not a nil ring, then without any further assumption (e.g. chain conditions) it is shown that a simple alternative ring is either associative or is a Cayley algebra over its center. If there is a commutator whose fourth power is not zero, this follows from earlier papers of the author. If the fourth power of every commutator is zero, then a result due to Levitzki shows that the nilpotent elements form an ideal. *Marshall Hall, Jr.*

Hua, Lo Ken. On semi-homomorphisms of rings and their applications in projective geometry. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 143–148 (1953). (Russian)

A semi-homomorphism of a ring R is a mapping $a \to a^\sigma$ of R into itself which preserves addition and is such that $(a^2)^\sigma = (a^\sigma)^2$, $(aba)^\sigma = a^\sigma b^\sigma a^\sigma$. It is shown that every semi-homomorphism of a ring without zero divisors is either a homomorphism or anti-homomorphism. As an application he reproves his earlier result [C. R. 1er Congrès Math. Hongrois, 1950, pp. 317–325, Akad. Kiadó, Budapest, 1952; these Rev. 15, 56] that a mapping of a projective line onto itself preserving harmonic ratios

$$(z_2 - z_4)^{-1}(z_2 - z_3)(z_1 - z_3)^{-1}(z_1 - z_4) = -1$$

is a generalized projective transformation

$$z' = (az^\sigma + b)(cz^\sigma + d)^{-1}.$$

Marshall Hall, Jr. (Columbus, Ohio).

Mori, Yoshiro. On the integral closure of an integral domain. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 249–256 (1953).

The main theorem stated in this paper is that if R is a Noetherian integral domain, then its integral closure $\bar{R}$ is normal (i.e., is an "endliche diskrete Hauptordnung" in the sense of Krull). This result, which answers in the affirmative a question raised by Krull some twenty years ago, would be very interesting if true. However it does not seem to be established except when R is a local domain in whose completion R^* the ideal (0) has no imbedded prime ideals.

The proof proceeds in four stages. First the author considers the case where R is a complete local domain, in which case the result follows easily from the known structure theory. Second, he considers the case where R is a local domain such that R^* has no nilpotent elements. Here there are some unsupported statements (e.g., on p. 252, that $\bar{R}^* \cap K = \bar{R}$, and on pp. 252–253, that q_{ij} is a symbolic power), but these gaps can be filled in. Third, he takes up the case of an arbitrary local domain by passing from the ring R^* to R^*/l^*, where l^* is the set of all nilpotent elements of R^*. This reduction can succeed, however, only when $R^* \cdot (0)$ is free from imbedded prime ideals, for while it is true that (see p. 253) K^*/l^*K^* is a quotient ring of R^*/l^*, it will not be the total quotient ring except when $R^* \cdot (0)$ is as stated above. The formal source of the error lies in the tacit assumption that the element π used later in the proof is not a zero divisor. Finally, the case of a general Noetherian domain is treated by reduction to the local case, but there seems to be some confusion in the notation and the reviewer was not able to follow this reduction.

In the last part of the paper the author considers a local domain R such that R^* has no nilpotent elements. Krull has proved [Math. Ann. 103, 450–465 (1930)] that when R is 1-dimensional this implies that $\bar{R}$ is a finite R-module. It is shown that the same conclusion is valid for arbitrary dimension, provided that one of several additional conditions holds, for example that R is of characteristic zero, or R is of characteristic $p > 0$ and its residue field F is finite over F^p.

I. S. Cohen (Cambridge, Mass.).

Nakayama, Tadasi. Wedderburn's theorem, weakly normal rings, and the semigroup of ring-classes. J. Math. Soc. Japan 5, 154–170 (1953).

L'auteur continue dans ce travail à développer ses idées visant à généraliser la théorie de Galois à des anneaux quelconques, en remplaçant les conditions de semi-simplicité par des conditions d'existence de bases finies [cf. Amer. J. Math. 74, 645–655 (1952); ces Rev. 14, 129]. Il commence par énoncer (prop. 1) une généralisation du théorème de Jacobson-Bourbaki: A étant un anneau ayant un élément unité, $\mathfrak{A}$ l'anneau des endomorphismes du groupe additif de A, il s'agit de caractériser, dans $\mathfrak{A}$, les sous-anneaux qui sont des commutants des anneaux de type C_R, C désignant un sous-anneau de A contenant l'unité, tel que A soit un C-module à droite ayant une base finie, C_R étant le sous-anneau de $\mathfrak{A}$ formé des homothéties $x \to xc$, où $c \in C$. Dans l'énoncé, la condition qu'un tel commutant contient l'élément unité n'est pas mentionnée (bien qu'utilisée dans la démonstration); en outre la démonstration de la nécessité de la condition de l'énoncé est incorrecte: elle repose en effet sur l'assertion (inexacte en général, comme il est bien connu) que deux bases d'un module à droite sur A ont le même nombre d'éléments (la démonstration esquissée en note p. 156 repose apparemment sur la relation $(\sum_i e_i A)\mathfrak{m} = \sum_i e_i \mathfrak{m}$ pour un idéal à droite $\mathfrak{m}$ de A, qui est évidemment inexacte). La validité des énoncés dépendant de la nécessité de la condition de la prop. 1 reste donc en doute, sans hypothèse supplémentaire sur les anneaux envisagés. Soit A_L le sous-anneau de $\mathfrak{A}$ formé des homothéties $x \to ax$ ($a \in A$); l'auteur dit que C est faiblement normal dans A si son commutant dans $\mathfrak{A}$ a une base finie sur A_L formée d'endomorphismes

semi-linéaires de A (considéré comme A-module à gauche); au cas où il existe une telle base formée d'endomorphismes linéaires de A, C est intérieurement faiblement normal. Dans ce cas, le commutant de C_R dans $\mathfrak{A}$ est le produit kroneckérien de K_R et de A_L sur le centre de A, K étant le commutant de C dans A; tout anneau S contenant A, ayant même élément unité que A et dont le centre contient le centre de A, est produit kroneckérien (sur C) de A et du commutant de K dans S (généralisation d'un théorème de Wedderburn); si C est simple (resp. primaire et satisfaisant à une condition minimale) il en est de même de A et de K. L'auteur discute enfin les relations entre ses résultats et certains théorèmes d'Azumaya [Nagoya Math. J. **2**, 119–150 (1951); ces Rev. **12**, 669], notamment en ce qui concerne une extension de la notion de groupe de Brauer aux classes d'anneaux faiblement normaux sur un anneau C donné. *J. Dieudonné* (Evanston, Ill.).

Lamprecht, Erich. Über I-reguläre Ringe, reguläre Ideale und Erklärungsmoduln. I. Math. Nachr. **10**, 353–382 (1953).

As the author himself acknowledges in a note added during the correction of the proofs, most of the concepts and results of this paper are not new. A "dual algebra" in the sense of the author is a "Frobenius algebra" as defined by Nakayama [Ann. of Math. (2) **40**, 611–633 (1939); these Rev. **1**, 3], that is an algebra on which there exists a linear form whose kernel does not contain any nonzero ideal; an "I-regular algebra" is what Nakayama [loc. cit.] calls a "quasi-Frobenius algebra", that is, an algebra in which the right annihilator of the left annihilator of any right ideal is that ideal, together with a corresponding condition for left ideals. These definitions are extended to rings [the "I-regular rings" are those called "dual rings" by I. Kaplansky, ibid. **49**, 689–701 (1948); these Rev. **10**, 7]; the author limits himself to rings with both chain conditions. As new results, one can mention a study of the ideal structure of a completely primary ring which is "I-regular", and the fact that such a ring is also "dual" if it is an algebra of finite rank; some interesting new examples of semi-primary "dual" rings are also given. *J. Dieudonné.*

Ikeda, Masatoshi. On absolutely segregated algebras. Nagoya Math. J. **6**, 63–75 (1953).

Let A be a finite-dimensional algebra with identity element over a field F. Then A is said to be absolutely segregated if, whenever φ is a homomorphism of a finite-dimensional algebra B onto A, there is an isomorphism ψ of A into B such that $\varphi\psi$ is the identity map on A. Equivalently, A is absolutely segregated if and only if all 2-dimensional cohomology groups of A in finite-dimensional two-sided A-modules are (0). The main result of the paper is that A, with radical N, is absolutely segregated if and only if the following two conditions are satisfied: (1) A/N is separable; (2) as a left A-module, N is a direct summand of a free A-module. The second condition is easily seen to be equivalent to the condition that N be an "M_0-module" (i.e., a projective module) in the sense that every A-homomorphism of an A-module onto N (regarded as a left A-module) has a right inverse. *G. Hochschild* (Urbana, Ill.).

Nagao, Hirosi. Note on the cohomology groups of associative algebras. Nagoya Math. J. **6**, 85–92 (1953).

A simplified proof is given of Ikeda's result, as described in the preceding review. Furthermore, the following generalization of a part of Ikeda's theorem is given: in the notation of the preceding review, denote by Q_{m-1} (for $m \geqq 2$) the right A-module on the tensor product (relative to the base field) $N \otimes A \otimes \cdots \otimes A$, with $m-2$ factors A, in which the A-operators are such that (for $n \in N$ and $a_i \in A$)

$$(n \otimes a_2 \otimes \cdots \otimes a_{m-1}) \cdot a_m = (-1)^m n a_2 \otimes a_3 \otimes \cdots \otimes a_m + \sum_{i=2}^{m-1} (-1)^{m-i-1} n \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_m.$$

Assume that A/N is separable. Then all m-dimensional cohomology groups of A are (0) if and only if Q_{m-1} is an M_0-module.

It is stated in a note at the end of the paper that Nakayama and Ikeda have proved that if, for some m, all m-dimensional cohomology groups of A vanish then A/N is separable. Consequently, the above gives a generalization of Ikeda's theorem characterizing the algebras with vanishing m-dimensional cohomology groups ($m \geqq 2$) in terms of the module Q_{m-1}. *G. Hochschild* (Urbana, Ill.).

Nakayama, Tadasi. On absolutely segretated algebras and relative 3-cohomology groups of an algebra. Nagoya Math. J. **6**, 177–185 (1953).

Results of Ikeda and Nagao [see the two preceding reviews] are generalized and refined. It is shown that if A is any algebra (of finite or infinite dimensionality) with vanishing 2-dimensional cohomology groups then every ideal of A (left or right) is an M_0-module. This result is obtained by considering, besides the usual cohomology groups of A, also the relative cohomology groups of A with respect to a left ideal. If, furthermore, A is algebraic (over the base field) and satisfies both chain conditions for left (or right) ideals, and if N denotes the radical of A, then the center of each simple component of the semisimple algebra A/N is separable over the base field.

Generalizations to higher cohomology dimensions [cf. the preceding review] have been obtained jointly by Ikeda, Nagao and Nakayama, and will be given in a subsequent joint paper by these authors. *G. Hochschild.*

Wolf, Paul. Galoissche Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. II. Math. Nachr. **10**, 233–238 (1953).

[For part I see Math. Nachr. **9**, 281–300 (1953); these Rev. **15**, 97.] If K is a Galois algebra over a field Ω then K determines (in an essentially unique manner) a Galois field K* over Ω. The Galois group $\mathfrak{G}^*$ of K*/Ω is a subgroup of the Galois group $\mathfrak{G}$ of K/Ω. Suppose now that K is also a Galois algebra over a subfield Ω_0 of Ω such that Ω is a Galois field over Ω_0. Let $\mathfrak{G}_0$ denote the Galois group of K/Ω_0. This is an extension of $\mathfrak{G}$ by means of the Galois group of Ω/Ω_0. The Galois group $\mathfrak{G}_0^*$ of the Galois field associated with K/Ω_0 is determined in terms of $\mathfrak{G}^*$ and the extension $\mathfrak{G}_0$. The systems $\mathfrak{B}$ introduced in the first part are studied. *R. Brauer* (Cambridge, Mass.).

Masuda, Katsuhiko. One-valued mappings of groups into fields. Nagoya Math. J. **6**, 41–52 (1953).

Soit G un groupe, Ω un corps, $G(\Omega)$ l'algèbre de groupe de G sur Ω, $\mathfrak{M}$ le $G(\Omega)$-module des applications de G dans Ω, la loi externe étant donnée par les translations à droite de G; l'auteur appelle algèbre de Galois relative à G toute algèbre $\mathfrak{K}$ obtenue en définissant sur $\mathfrak{M}$ une multiplication qui en fait une algèbre sur Ω (non nécessairement associative), et pour laquelle les translations à droite de G sont des automorphismes de la structure d'algèbre. Pour toute

représentation matricielle $\sigma \to D(\sigma) = (d_{ij}(\sigma))$ de G par des matrices sur Ω, $D = (d_{ij})$ est une matrice à éléments dans $\mathfrak{M}$. Soit $\Gamma = (D_\alpha)$ une famille de représentations matricielles de G sur Ω deux à deux non équivalentes, telle que $D_\alpha(\sigma) \otimes D_\beta(\sigma) = P_{\alpha\beta}^{-1}(D_{\gamma_1}(\sigma) \oplus \cdots \oplus D_{\gamma_k}(\sigma))P_{\alpha\beta}$ ($P_{\alpha\beta}$ étant une matrice fixe sur Ω) pour tout couple (α, β); alors, si $\mathfrak{K}$ est une algèbre de Galois, le produit tensoriel $D_\alpha \otimes D_\beta$ (pris au sens de la multiplication dans $\mathfrak{K}$) est égal à $P_{\alpha\beta}^{-1}(D_{\gamma_1} \oplus \cdots \oplus D_{\gamma_k})P_{\alpha\beta}C_{\alpha\beta}$, où $C_{\alpha\beta}$ est une matrice sur Ω; les coefficients d_{ij} des D_α forment donc une sous-algèbre $\mathfrak{D}_\Gamma$ de $\mathfrak{K}$. Si on suppose d'abord que G est fini et qu'on prend pour Γ la famille des constituants indécomposables de la représentation régulière de G sur Ω, on a $\mathfrak{D}_\Gamma = \mathfrak{K}$, et les $C_{\alpha\beta}$ déterminent donc la multiplication dans $\mathfrak{K}$; l'auteur caractérise ces "systèmes de facteurs" $C_{\alpha\beta}$ dans le cas général, ainsi que dans le cas où $\mathfrak{K}$ est associative, commutative et semi-simple, et étudie les relations entre "systèmes de facteurs" de G et de sous-groupes de G satisfaisant certaines conditions. G étant de nouveau arbitraire, pour tout automorphisme ρ de $\mathfrak{D}_\Gamma$ permutant avec les opérateurs de G, on a $D_\alpha{}^\rho = D_\alpha C_\alpha$, où C_α est une matrice sur Ω, et le groupe de ces automorphismes est isomorphe à celui des systèmes (C_α) de matrices (groupe associé à Γ); certains de ces groupes sont isomorphes à des centralisateurs de sous-groupes de G (lorsque G est fini). Lorsque G est compact, l'auteur prouve le théorème de Tannaka sous la forme suivante: prenant pour Γ le système des représentations unitaires irréductibles de G, G est isomorphe au sous-groupe du groupe associé à Γ, formé des systèmes de matrices unitaires (C_α), et muni de la topologie produit des topologies de groupe unitaire. *J. Dieudonné* (Evanston, Ill.).

Kolchin, E. R. Galois theory of differential fields. Amer. J. Math. 75, 753–824 (1953).

The author extends the Galois theory of Picard-Vessiot extensions [Ann. of Math. (2) 49, 1–42 (1948); Proc. Amer. Math. Soc. 3, 596–603 (1952); these Rev. 9, 561; 14, 241] to "strongly normal" (ordinary and partial) differential field extensions G of an arbitrary differential field F of characteristic 0 subject to the restrictions that they have the same constant field C as F and that they are finitely generated and of finite transcendence degree over F. The modified concept of strong normality used in this paper permits an algebraico-geometric characterization of those subgroups of the automorphism group of G over F which will appear in the Galois-correspondence to be established.

Following the procedure in algebraic geometry the theory of specializations is extended to differential fields and the existence of a universal extension F^* of F is proved. In the following every occurring extension G of F is tacitly assumed to be a subfield of F^*, and every isomorphism σ of G is assumed to be an isomorphism into F^*. An isomorphism σ' of G is defined to be a specialization of σ if $(\sigma'\alpha)_{\alpha \in G}$ is a specialization of $(\sigma\alpha)_{\alpha \in G}$. An isomorphism σ of G over F is called strong if $G\langle\sigma G\rangle = G\langle C\sigma\rangle = \sigma G\langle C\sigma\rangle$ holds where $C\sigma$ denotes the constant field of $G\langle\sigma G\rangle$. Specializations of strong isomorphisms are strong, too. If C^* stands for the field of constants of F^*, the mapping assigning to each automorphism of $G\langle C^*\rangle$ over $F\langle C^*\rangle$ its restriction to G is one-to-one onto the set $\mathfrak{G}^*$ of all strong isomorphisms of G over F; hence, by a natural identification a multiplication can be defined in $\mathfrak{G}^*$ which makes $\mathfrak{G}^*$ to an over-group ~~a subgroup~~ of the full automorphism group $\mathfrak{G}$ of G over F. Now a subset $\mathfrak{M}^*$ of $\mathfrak{G}^*$ is called an irreducible set in $\mathfrak{G}^*$ if it is the set of all specializations of one of its elements, a strong isomorphism σ^* generating $\mathfrak{M}^*$ in this way is called a generic element of $\mathfrak{M}^*$, and the transcendence degree $\partial^0 G\langle\sigma^* G\rangle/G$ which is independent of the choice of σ^* is called the dimension of $\mathfrak{M}^*$. A subset $\mathfrak{M}^*$ of $\mathfrak{G}^*$ is called an algebraic set in $\mathfrak{G}^*$ if it is the union of finitely many irreducible sets in $\mathfrak{G}^*$. The maximal irreducible sets contained in an algebraic set are called its components. Since $\mathfrak{M}^* \neq (0)$ implies $\mathfrak{M}^* \cap \mathfrak{G} \neq (0)$, and $\mathfrak{M}^* \neq \mathfrak{N}^*$ for two algebraic sets $\mathfrak{M}^*$, $\mathfrak{N}^*$ implies $\mathfrak{M}^* \cap \mathfrak{G} \neq \mathfrak{N}^* \cap \mathfrak{G}$ the algebraico-geometric structure in $\mathfrak{G}^*$ induces a similar structure in $\mathfrak{G}$. Now an algebraic group $\mathfrak{H}$ in $\mathfrak{G}$ is defined as a subgroup of $\mathfrak{G}$ which is also an algebraic set. The properties of algebraic groups are, of course, very much like those of algebraic matric groups.

The author introduces three kinds of normality for an extension G of a differential field F. The extension G is defined to be strongly normal if every isomorphism of G over F is strong, normal if the set of all automorphisms of G over F is abundant, and weakly normal if all invariants of the group of all automorphisms of G over F belong to F. Strong normality implies normality, but not vice versa (as it is shown by an example). Normality implies weak normality; whether the converse is true is still an open question (but it is surely false if the restriction is dropped that G should have the same field of constants as F as demonstrated by an example). If G is a strongly normal extension over F, its group $\mathfrak{G}$ of all automorphisms over F is algebraic, there is a one-to-one Galois correspondence between the set of all intermediate differential fields F_γ and the set of all algebraic subgroups $\mathfrak{G}_\gamma$ of $\mathfrak{G}$ and the transcendence degree of G over F_γ equals the dimension of $\mathfrak{G}_\gamma$. Weak normality of an intermediate F_γ over F implies already strong normality, and this is equivalent to $\sigma F_\gamma \subseteq F_\gamma$ for every automorphism $\sigma \in \mathfrak{G}$ and likewise equivalent to the normality of the corresponding subgroup $\mathfrak{G}_\gamma$ of $\mathfrak{G}$. If $\mathfrak{G}_\gamma$ is normal, $\mathfrak{G}/\mathfrak{G}_\gamma$ is isomorphic to the group of all automorphisms of F_γ over F.

Let G be an algebraic function field of one variable over an algebraically closed field F. If G is a weakly normal differential field extension of F, it is strongly normal over F. If, additionally, G has genus 0, it can be obtained from F by a differential field adjunction of an element $\alpha \in G$ either satisfying $\delta_i\alpha \in F$ for all derivations δ_i of $\mathfrak{G}$ (in which case α will be called primitive over F) or satisfying $\alpha\delta_i\alpha \in F$ (then α is called exponential over F), and if G has genus 1, by adjunction of an $\alpha \in G$ satisfying

$$(\delta_i\alpha)^2 = a_i^2(4\alpha^3 - g_2\alpha - g_3),$$

where $a_i \in F$; $g_2, g_3 \in C$ and $4x^3 - g_2x - g_3$ must have only simple zeros (then α is called weierstrassian over F). The genus of G cannot exceed 1 since then the group of all automorphisms of G over F is finite and hence G cannot be weakly normal over F. If F is only relatively algebraically closed in G the situation remains the same for genus 0, but in the case of genus 1 a weierstrassian element α can be found so that in general only an algebraic extension of $F\langle\alpha\rangle$ is equal to G. Now let α be an element of one of those three types. If α is transcendental over F, then $F\langle\alpha\rangle$ has transcendence degree 1 over F. Here $F\langle\alpha\rangle$ is a Picard-Vessiot extension in the first two cases, but in the third one only if α is also algebraic over F. If a Picard-Vessiot extension H of F is contained in an extension K of F obtained by successive adjunction of finitely many elements being algebraic, primitive, exponential, or weierstrassian over the preceding extension in the constructing of K, then H is a Liouvillian extension of F. *A. Jaeger* (Cincinnati, Ohio).

Cohn, Paul, and Mahler, Kurt. On the composition of pseudo-valuations. Nieuw Arch. Wiskunde (3) 1, 161–198 (1953).

This paper is a continuation of the senior author's work on pseudo-valuations, that is, on non-negative real-valued functions W on rings $R=\{a, b, \cdots\}$ for which (i) $W(0)=0$, (ii) $W(a-b)\leqq W(a)+W(b)$, and (iii) $W(ab)\leqq W(a)W(b)$ [if $W(a-b)\leqq \max\{W(a), W(b)\}$, then W is called non-archimedean]. It is shown that each real function φ on R satisfying (i) defined by

$$W_\varphi(a)=\sup\{W(a)\mid W\in\Omega_R,\ W\leqq\varphi\},$$
$$\bar{W}_\varphi(a)=\sup\{\bar{W}_\varphi(a)\mid W\in\Omega_R,\ W\text{ non-archimedean},\ W\leqq\varphi\},$$

where Ω_R is the set of all pseudo-valuations on R, two pseudo-valuations for which, using $\leqq$ in the obvious manner, $0\leqq\bar{W}_\varphi\leqq W_\varphi\leqq\varphi$, $\bar{W}_\varphi$ is non-archimedean. Equivalent definitions for these pseudo-valuations which are majorized by φ are given in terms of the multiplicative and additive representations of the argument a. Next four types of binary operations with values in Ω_R are defined on Ω_R. In the sequel all processes indicated are taken for all possible decompositions $a=x_1y_1+\cdots+x_ny_n$, where $x_i, y_i\in R$:

$$W_1\cdot W_2(a)=\inf\max_i\{W_1(x_i)W_2(y_i)\},$$
$$W_1\odot W_2(a)=\inf\textstyle\sum_i W_1(x_i)W_2(y_i),$$
$$W_1\times W_2(a)=\inf\max_i\{W_1(x_i), W_2(y_i)\},$$
$$W_1\otimes W_2(a)=\inf\textstyle\sum_i(W_1(x_i)+W_2(y_i)).$$

It follows that these operations are commutative and associative; $W_1\cdot W_2$ and $W_1\times W_2$ are non-archimedean. Observations for special cases lead the authors to seek analogues to operations for ideals. Define $W_1\sim W_2$ to mean that for any sequence $a_n\in R$, $W_1(a_n)\to 0$ if and only if $W_2(a_n)\to 0$, and let $W_1\subset W_2$ mean that to every $\epsilon_1>0$ there corresponds an $\epsilon_2>0$ such that $W_1(a)<\epsilon_1$ holds whenever $W_2(a)<\epsilon_2$. Then many relations between ideals involving "$=$" and "$\supseteq$" lead to similar relations between bounded non-archimedean pseudo-valuations involving "$\sim$" and "$\subset$". For example, $W_1\times W_2\subset(W_1+W_2)\times(W_1\cdot W_2)$ holds as analogue to $\mathfrak{ab}\supseteq(\mathfrak{a}\cap\mathfrak{b})(\mathfrak{a}+\mathfrak{b})$ for ideals $\mathfrak{a}$, $\mathfrak{b}$. However, this type of formal correspondence is shown to break down for distributive laws in special rings. Furthermore, the authors set up complete multiplication tables for each of the four operations applied to finite fields of algebraic numbers and the rings of all algebraic integers in the latter. The verification of the tables uses the senior author's former decomposition theorems of pseudo-valuations into $\sim$-equivalent sums of valuations and residue-class pseudo-valuations [see K. Mahler, Acta Math. 67, 51–80, 283–328 (1936)].

O. F. G. Schilling (Chicago, Ill.).

Verhoeff, J. On pseudo-convergent sequences. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 401–404 (1953).

Let K be a field with a non-Archimedean absolute value $|\ |$. It is proved by elementary considerations that if $f(x)\in K[x]$ is a polynomial of degree $n>0$, and if $\{a_i\}$ is a pseudo-convergent sequence in K, then there exists a real number $c>0$ and an integer k, $1\leqq k\leqq n$, such that $|f(a_{i+1})-f(a_i)|=c|a_{i+1}-a_i|^k$ for sufficiently large i. This implies that $\{f(a_i)\}$ is likewise pseudo-convergent, a result proved by Ostrowski [Math. Z. 39, 269–404 (1934)] using the theory of algebraic extensions of valued fields. In addition, a counter-example is given to a lemma of Loonstra [same Proc. 45, 913–917 (1942); these Rev. 6, 144].

I. S. Cohen (Cambridge, Mass.).

Theory of Groups

Lyapin, E. S. Associative systems of all partial transformations. Doklady Akad. Nauk SSSR (N.S.) 88, 13–15; errata 92, 692 (1953). (Russian)

Etant donné un système associatif ($=$demigroupe), c'est-à-dire, un ensemble $\mathfrak{U}$ muni d'une multiplication partout définie associative, on dira que $\mathfrak{U}$ est de classe Σ_1 lorsque les quatre conditions suivantes sont remplies: 1) $\mathfrak{U}$ admet un élément nul 0; 2) le produit de deux idempotents différents est toujours 0; 3) pour tout $a\in\mathfrak{U}$ il existe deux idempotents e, j tels que $ea=aj=a$; 4) quels que soient les idempotents e, j il existe $a\neq 0$ tel que $ea=aj=a$. On dira que $\mathfrak{U}$ est de classe Σ_2 lorsque, quels que soient $x, y\in\mathfrak{U}$, $xy=x$ ou $xy=y$. Ces deux définitions ainsi que celle d'idéal "dense dans $\mathfrak{U}$" permettent d'énoncer les deux caractérisations abstraites suivantes: Pour que $\mathfrak{U}$ soit isomorphe au demigroupe constitué par toutes les relations biunivoques d'un certain ensemble dans lui-même (la multiplication étant identifiée à la composition des relations), il faut et il suffit que $\mathfrak{U}$ possède un idéal dense de classe Σ_1. Pour que $\mathfrak{U}$ soit isomorphe au demigroupe constitué par toutes les applications d'un certain ensemble dans lui-même, il faut et il suffit que $\mathfrak{U}$ possède un idéal dense de classe Σ_2. Ce dernier résultat n'est qu'une modification d'un résultat analogue de Malcev [Mat. Sbornik N.S. 31(73), 136–151 (1952); ces Rev. 14, 349].

J. Riguet (Paris).

Baer, Reinhold. The hypercenter of a group. Acta Math. 89, 165–208 (1953).

In this paper, the upper hypercentre of a group is the terminal member of its (transfinite) upper central series, and an upper hypercentral subgroup is a normal subgroup contained in the hypercentre. The problem is the intrinsic characterization of these objects. Key notions are those of a finitely reducible (f.r.) subgroup, and of a fractionally finitely reducible (f.f.r.) subgroup. (The latter are called locally f.r. by the author, but this conflicts with the systematic meaning of that adverb.) A normal subgroup N of a group G is thought of as a group with operators, namely the inner automorphisms of G; and N is f.r. if it is not 1, and if every element not 1 in it is distinguished from 1 in some operator homomorphism of N onto a finite operator simple group. N is f.f.r. if every operator homomorph of it has an operator subgroup which is f.r. Typical of the effect of these conditions is the result: A finitely generated f.f.r. p-group is finite. Upper hypercentral subgroups are f.f.r.

An upper hypercentral subgroup N of G satisfies certain commutativity conditions. Thus an element of N and an element of G commute if they are of relatively prime finite orders; and, more generally, if x in N is of order m, and g is any element of G, then x and g^{m^r} commute for large enough r. In the converse direction, if N is f.f.r. and periodic, and if to every element x of prime order p in N and to every element g of G corresponds an r such that x and g^{p^r} commute, then N is upper hypercentral. If we make the last assumption not only for N and G, but also for N/M and G/M whenever M is a normal subgroup of G contained in N, then we can drop the assumption that N is periodic. Again, if N is upper hypercentral, and if T is any subgroup of G not containing N, then T is distinct from its normaliser in NT. The converse is true provided that N is f.f.r.

Further results include the following. Finitely many elements of finite order in an upper hypercentral subgroup generate a finite group. A periodic group is upper nilpotent (its own upper hypercentre) if and only if it is a direct

product of f.f.r. primary subgroups. A finitely generated group is upper nilpotent only if it is nilpotent of some finite class; and each of the following statements is necessary and sufficient for this: (i) G is f.f.r., and satisfies the maximum condition for subgroups, and its maximal subgroups are normal; (ii) G is f.f.r., and satisfies the maximal condition for normal subgroups, and its finite quotient groups are nilpotent. *Graham Higman* (Manchester).

Baer, Reinhold. Das Hyperzentrum einer Gruppe. II. Arch. Math. **4**, 86–96 (1953).

Observing that various properties which characterize the (upper) hypercentre of a finite group cease to be equivalent for infinite groups, the author begins by listing five postulates that must be satisfied by any reasonable definition of hypercentral subgroups (normal subgroups contained in the hypercentre). These say, roughly, that the class defined must specialise correctly for finite groups, must relativise in the natural way in subgroups and factor groups, and must be closed under formation of products. This leads him to adopt the following definition, which is the weakest consistent with his postulates: The normal subgroup N of G is hypercentral if whenever U is a subgroup of G and V a normal subgroup of U of finite index, $U\cap N\cdot V/V$ is a hypercentral subgroup of the finite group U/V.

Other conditions considered include upper hypercentrality (cf. the preceding review) and the two following:

(N) N is an (N)-subgroup of G if every subgroup U of G not containing N is distinct from its normaliser in UN.

(M) N is an (M)-subgroup of G if whenever U is a maximal subgroup of a subgroup V of G, then either $V\cap N\subset U$ or U is normal in V.

Then these implications hold:

$$\text{upper hypercentral}\rightarrow(N)\rightarrow(M)\rightarrow\text{hypercentral},$$

and upper hypercentral is equivalent to hypercentral and f.f.r. (cf. the preceding review). *Graham Higman.*

Higman, D. G. Focal series in finite groups. Canadian J. Math. **5**, 477–497 (1953).

If S is a subgroup of a finite group G, denote by μS the subgroup generated by all quotients $x^{-1}y$ where x, y belong to S and are conjugate in G. The focal series of S in G is the series $S, \mu S, \mu^2 S, \cdots$. More generally, a focal chain in G is a series $S_1, S_2, \cdots$ of subgroups such that $S_i\supset S_{i+1}\supset\mu S_i$. If S is the first and H the last member of a focal chain, H is said to be chained to S in G. If 1 is chained to S in G, S is called a hyperfocal subgroup of G. The object of introducing these concepts is to obtain conditions under which a factor group S/H of a subgroup of G can be realised as a factor group of G (i.e. under which there is a normal subgroup N of G such that $G=NS$ and $H=N\cap S$). For instance, it is sufficient for this that (i) H is chained to S in G, and (ii) the indices of H in S and of S in G are coprime. In particular, if a Sylow subgroup of G (or more generally a subgroup whose order and index are coprime) is hyperfocal, it has a normal complement; if the subgroup is nilpotent, the converse is also true. Because the focal series of a Sylow subgroup in G is the same as its focal series in its normaliser, the direct half of this result generalises the classical theorem of Burnside about a Sylow subgroup which lies in the centre of its normaliser. A crucial intermediate result, proved by the aid of Schur's transfer, is the following: Let H be chained to S in G, let π be a set of primes including all prime divisors of the index of H in S, and let $G/P(\pi)$ be the maximum π-factor of G; then every prime factor of the index of $H\cap P(\pi)$ in $S\cap P(\pi)$ divides the index of $S\cap P(\pi)$ in $P(\pi)$. *Graham Higman* (Manchester).

Kulikov, L. Ya. On direct decompositions of groups. Ukrain. Mat. Žurnal **4**, 230–275, 347–372 (1952). (Russian)

Let G be a torsion-free abelian group which is completely decomposable in the sense of being a direct sum of groups of rank one. Is every direct summand of G likewise completely decomposable? This question was raised by Baer [Duke Math. J. **3**, 68–122 (1937)] and was answered by him in the affirmative under the extra assumption that the isomorphism types of the components satisfy a certain descending chain condition. The main new result of the present paper is to answer the question in the affirmative in case G is countable. This is the end product of a long investigation, many aspects of which are of independent interest.

Let $G=\sum A_i$, $i\in I$, be a direct sum decomposition of a group G (not necessarily abelian) and let Ω be a set of endomorphisms of G. The index set I is topologized by declaring a subset J to be closed in case $\sum A_j$, $j\in J$, is invariant under Ω. A critical question is whether I thus topologized is a Kolmogoroff space (distinct points have distinct closures). If it is, we get a natural partial ordering $L(I,\Omega)$ of I by setting $i\leqq j$ when i is in the closure of j. In case Ω is the set of all normal endomorphisms of G (i.e., endomorphisms commuting with inner automorphisms), the Kolmogoroff condition is equivalent to the following: for no two distinct summands is it the case that each admits a non-trivial homomorphism into the center of the other. Theorem 10.3 reads as follows: Suppose (1) $G=\sum A_i$ has the Kolmogoroff condition relative to the set of all normal idempotent endomorphisms of G, (2) for each i, any two decompositions of A_i admit centrally isomorphic refinements, (3) either (a) G is countable or (b) $L(I,\Omega)$ satisfies the descending chain condition; then any two direct decompositions of G admit centrally isomorphic refinements. (Two decompositions are centrally isomorphic if one is carried into the other by an automorphism $x\rightarrow x'$ such that $x'x^{-1}$ is always central.)

In the application to a torsion-free completely decomposable abelian group, one forms $G=\sum A_i$ by grouping together in one summand all isomorphic summands of rank one. Conditions (1) and (2) above are then verified; condition 3(b) coincides with the hypothesis used by Baer.

The paper concludes by extending the result to completely decomposable abelian groups which are not necessarily torsion-free. The main fact needed is the author's earlier theorem [Mat. Sbornik N.S. **16(58)**, 129–162 (1945); these Rev. **8**, 252] that any subgroup of a direct sum of cyclic groups is cyclic. *I. Kaplansky* (Chicago, Ill.).

Chehata, C. G. Commutative extension of partial automorphisms of groups. Proc. Glasgow Math. Assoc. **1**, 170–181 (1953).

Suppose that a is an isomorphism of the subgroup A of the group G onto the subgroup A^a of G, and that b is an isomorphism of the subgroup B of G onto the subgroup B^b of G. If furthermore $(A\cap B)^a=A^a\cap B$, $(A\cap B)^b=A\cap B^b$, $(A^a\cap B)^b=A^a\cap B^b=(A\cap B^b)^a$, and if $g^{ab}=g^{ba}$ for every g in $A\cap B$, then there exists a group E containing G and automorphisms a^*, b^* of E which commute and which induce a and b in A and B respectively. The proof of this theorem is effected by means of a construction based on the existence of certain free products with amalgamated subgroups. *R. Baer* (Urbana, Ill.).

Eckmann, Beno. Cohomology of groups and transfer. Ann. of Math. (2) **58**, 481–493 (1953).

The author considers the following situation. Let $B \subset A$ be groups, R_A the integral groupring of A. To every B-module J the author assigns an A-module Ψ (namely the A-module of B-homomorphisms of R_A into J). The first result of the paper is that the resulting homomorphism of $H^p(B; J) \rightarrow H^p(A; x)$ is an isomorphism onto. If in addition the index of B in A is finite, and J is an A-module, an A-homomorphism $t: \Psi \rightarrow J$ can be constructed. Hence under these conditions a homomorphism, again denoted by t, exists which maps $H^p(B, J) \xrightarrow{t} H^p(A, J)$. Here t is compared with the restriction homomorphism $Q: H(A, J) \rightarrow H(B, J)$, and it is shown that tQ coincides with multiplication by n: $tQ = M_n$. The homomorphism t generalizes the classical notion of transfer in the sense that for $p=1$, t is the dual of the classical transfer of A into B/commutator subgroup of B. Several applications of these notions are given. For instance, it is shown that if A has a free subgroup of index n, then $nH(A, J) = 0$. Another application is that, if M_n is an automorphism of $H(B, J)$, then Q is an isomorphism into. This paper gives a systematic development of the cohomology theory of complexes over rings and is essentially self-contained. The author acknowledges in the introduction some overlapping of his results with unpublished results of Tate and Artin.

R. H. Bott (Ann Arbor, Mich.).

Bargmann, V. On unitary ray representations of continuous groups. Ann. of Math. (2) **59**, 1–46 (1954).

A unitary ray is a class of operators on a Hilbert space of the form $[\alpha U: \alpha \text{ complex}]$, where U is a fixed unitary operator. These rays form a topological group with respect to multiplication via the multiplication of unitary representatives and the obvious strong topology. The present paper treats continuous representations of topological, and especially Lie, groups by unitary rays, using in part techniques originating in Schur's treatment of projective representations of finite groups. Results of H. Weyl [Gruppentheorie und Quantenmechanik, 2. Aufl., Hirzel, Leipzig, 1931] are extended, including his theorem that a finite-dimensional representation of a simply connected group always arises from an ordinary unitary representation in the direct fashion. In any case there exists in a neighborhood of the group identity a (real, continuous) "local exponent" $\xi(a, b)$ such that for a suitable choice of unitary representatives U_a for a given unitary ray representation, $U_a U_b = \exp(i\xi(a, b)) U_{ab}$. Such local exponents form a linear space, which becomes finite-dimensional when local exponents defining unitarily equivalent ray representations are identified; its dimension is in fact not greater than $\frac{1}{2}n(n-1)$ when G is an n-dimensional Lie group. The upper bound is attained only for abelian groups.

When G is a Lie group, the local exponent can be chosen so as to be differentiable (by an application of an argument due to Iwasawa). This leads to a "canonical" local exponent and thence to an "infinitesimal" exponent defined on the Lie algebra of the group. A simple necessary and sufficient condition that a 2-form be an infinitesimal exponent is derived. Applications are made to special classes of groups, including the abelian (previously treated by H. Weyl, loc. cit.), pseudo-orthogonal groups, and the Galilean group. It results also that every local exponent of a semi-simple group is equivalent to the zero exponent, i.e. arises from a local unitary representation.

I. E. Segal.

Gerstenhaber, Murray. On the algebraic structure of discontinuous groups. Proc. Amer. Math. Soc. **4**, 745–750 (1953).

Let S be a connected and locally connected Hausdorff space, and let G be a group of homeomorphisms operating on S. Assume that G has a fundamental domain $D \subset S$. Denote by F the set of all elements $g \in G$ such that $g\bar{D} \cap \bar{D}$ is not empty. Assume that F is finite and that the set $FD = \bigcup_{g \in F} gD$ contains a neighbourhood of D. It is known that the elements of F generate G. Let $g_1, \cdots, g_n$ be the elements of F. The relations of the form $g_i g_j g_k^{-1} = 1$ satisfied by these elements and those following from them are called the local relations (relative to D) of G. Let H be the group generated by $g_1, \cdots, g_n$ and having these local relations as defining relations. It is called the local universal covering group of G. There is a natural homomorphism φ of H onto G. The author proves the existence of a connected covering space T of S with the following properties: H admits a representation as a group of homeomorphisms of T having a fundamental domain E which, by the covering mapping ψ of T onto S, is mapped topologically onto D. For any point $y \in T$ and any $h \in H$, $\psi(hy) = \varphi(h)\psi(y)$. If, in particular, S is simply connected and thus $T = S$, it follows that H and G are isomorphic and consequently all the relations of G are local. [The author's concluding remark: "In theory, at least, it (the main theorem of the paper) gives the structure of most of the common discontinuous groups of complex-analytic homeomorphisms operating on the upper half-plane, and in fact readily gives the structure of those groups whose fundamental domains are of a simple nature, such as the modular group" might mislead readers. The structures of these groups are known; they were found by Poincaré, Klein and Fricke just by determining the local relations belonging to suitable fundamental polygons. See for instance P. Fatou, Fonctions automorphes (vol. II of P. Appell and E. Goursat, Théorie des fonctions algébriques, 2nd ed., Gauthier-Villars, Paris, 1930), Chapt. XIV.]

W. Fenchel (Copenhagen).

★Maak, Wilhelm. Darstellungstheorie unendlicher Gruppen und fastperiodische Funktionen. Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 1, 16. Band I. Algebra und Zahlentheorie. 1. Teil. B. Algebra. Heft 7, Teil I. B. G. Teubner Verlagsgesellschaft, Leipzig, 1953. 26 pp. DM 3.00.

This is a rapid survey, with at most brief indications of proofs, of the basic facts concerning almost periodic functions on a group and their relationship to the finite-dimensional bounded representations of the group. The general theory including some recent contributions of the author is outlined in the first twenty sections (part A). Section 18 treats the equivalence between the theory and the Haar-Peter-Weyl theory of continuous functions on a compact group, but the almost periodic point of view is maintained. Part B (sections 21–23) is concerned with various generalizations of the theory; Weyl almost periodic functions, ω almost periodic functions, almost periodic functions with values in a topological vector space and almost periodic functions on semi-groups. The final part C (sections 24–30) deals with various specializations and applications; Bohr almost periodic functions, spherical harmonics, compact groups, the duality theorems of Pontrjagin and Tannaka, etc.

G. W. Mackey (Cambridge, Mass.).

Harish-Chandra. Representations of semisimple Lie groups. II. Trans. Amer. Math. Soc. **76**, 26–65 (1954).

This is a continuation of part I [same Trans. **75**, 185–243 (1953); these Rev. **15**, 100]. One of the main results of the present paper (theorem 4) has been announced earlier [Proc. Nat. Acad. Sci. U. S. A. **37**, 691–694 (1951); these Rev. **13**, 107]. It asserts that every irreducible quasi-simple (in the sense of the author) representation of a complex semi-simple Lie group G can be deduced from certain multiplier representations of G by taking quotient spaces and applying the operation of infinitesimal equivalence. The author's proof of this result is quite difficult and lengthy and involves a detailed analysis of the algebraic structure of the associative enveloping algebra B generated by the Lie algebra of G. In the course of this analysis many interesting results about the structure of B are obtained, in particular, properties of the ideals and representations of B. This together with earlier results of the author constitutes a very penetrating analysis of B.

On the other hand, if one is mainly interested in the representations of the group G it seems possible that a different approach would be better. For example in the case where G is a simple non-exceptional complex Lie group, Gelfand and Naimark [Trudy Mat. Inst. Steklov. **36** (1950); these Rev. **13**, 722] have studied the irreducible unitary representations by entirely different methods. They do not use the associative algebra B (not even the Lie algebra of G), only properties of the group in the large; but they also do not obtain any results about B. However, for the unimodular group (at least in the 2×2 case) the results of Gelfand and Naimark [loc. cit. and Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 495] are more complete and explicit as far as the irreducible unitary representations of the group are concerned. This together with the recent work on induced representations suggests that if one is mainly interested in the representations of the group itself, methods which use the group in the large throughout should be considered as an alternative to the present author's algebraic analysis of B.

F. I. Mautner (Baltimore, Md.).

Bruhat, François. Sur les représentations induites des groupes de Lie. C. R. Acad. Sci. Paris **237**, 1478–1480 (1953).

Let G be a Lie group and Γ a closed subgroup of G. Let a and b be two one-dimensional continuous unitary representations of Γ and denote by U^a and U^b the corresponding induced representations of G [Mackey, Ann. of Math. (2) **55**, 101–139 (1952); these Rev. **13**, 434]. The author studies the intertwining operators between U^a and U^b. They can be identified with certain distributions (in the sense of L. Schwartz) on G; this implies a generalization of one of the results of Mackey (loc. cit.) to the case where the double cosets of Γ in G need not satisfy Mackey's assumptions. The author mentions without giving details that he can also generalize certain irreducibility criteria. Reviewer's remark: It would be very interesting to see whether one can use this approach to generalize the Frobenius reciprocity theorem and obtain results for the important case where the double cosets of Γ in G are ergodic. It is often also necessary to consider induced representations U^a, U^b, $\cdots$ where a and b are not one-dimensional, but arbitrary unitary representations of Γ.

F. I. Mautner (Baltimore, Md.).

Dynkin, E. B. Construction of primitive cycles in compact Lie groups. Doklady Akad. Nauk SSSR (N.S.) **91**, 201–204 (1953). (Russian)

The author states a slight strengthening of a theorem of Hopf [Ann. of Math. (2) **42**, 22–52 (1941); these Rev. **3**, 61] and the reviewer [ibid. **42**, 1091–1137 (1941); these Rev. **3**, 143]: If $x_1, \cdots, x_r$ form a basis for the group of primitive elements of the homology group $H(G)$ of the compact Lie group G (integral cycles, rational homologies), then the point v_0 and the elements $x_{i_1}\cdot x_{i_2}\cdot \cdots \cdot x_{i_k}$, $1\leq i_1<\cdots<i_k\leq r$ (Pontryagin products) form a basis for $H(G)$. Next he proves a theorem stating that certain mappings of a special class of polyhedra produce primitive homology elements. Maps of this kind are then indicated for the classical groups in order to construct bases for the primitive elements. The elements so obtained coincide with those constructed by Pontryagin [Mat. Sbornik N.S. **6**(**48**), 389–422 (1939); these Rev. **1**, 259]. The maps are of the following type: Let K be a 1-parameter subgroup of G, Z the e-component of its centralizer, W the coset space G/Z. Map $W\times K$ into G by sending (X, h) into xhx^{-1}, where $h \in K$, $X \in W$, and $x \in X$; for each $z \in H(W)$ of positive dimension the image of $z\otimes K$ is then primitive. This construction appears also in a paper by E. Stiefel [Colloques Internat. Centre Nat. Recherche Sci., no. 12, Paris, 1949, pp. 97–101; these Rev. **11**, 499].

H. Samelson.

Yamabe, Hidehiko. A generalization of a theorem of Gleason. Ann. of Math. (2) **58**, 351–365 (1953).

The results of this paper are summarized in the following theorems. Theorem 3: A locally compact group with a neighborhood of the identity which does not contain any non-trivial subgroup is a Lie group. This theorem generalizes Gleason's in that the given group is not required to be finite-dimensional. Combined with a previous result by the author [Ann. of Math. (2) **58**, 48–54 (1953); these Rev. **14**, 948] similarly generalizing work of Montgomery and Zippin [ibid. (2) **56**, 213–241 (1952); these Rev. **14**, 135], this establishes the following theorem. Theorem 4: A locally compact connected group with a neighborhood of the identity which does not contain any non-trivial normal subgroup is a Lie group. Combined with previous work of Iwasawa and of Gleason, and resting heavily on results of Kuranishi [Proc. Amer. Math. Soc. **1**, 372–380 (1950); Nagoya Math. J. **1**, 71–81 (1950); these Rev. **12**, 77, 391] this gives the final theorem. Theorem 5: Every locally compact group is a generalized Lie group (in the sense of Iwasawa and of Gleason). For finite-dimensional groups this theorem had been proved by Montgomery and Zippin [loc. cit.].

The methods of this paper continue those of the author's preceding paper with a number of interesting new devices applied to the "tangent space" introduced by Gleason to study the family of one-parameter subgroups of a group with no small subgroups. The author succeeds in showing that this tangent space is finite-dimensional without having had to assume that the original group is finite-dimensional.

L. Zippin (Flushing, N. Y.).

Becker, Horst. Über einen Satz der Darstellungstheorie topologischer Gruppen. Wissensch. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **2**, no. 5, 61–66 (1953).

Results concerning the relation between representations of a locally compact group and of its L_1-algebra, of whose largely known character the author is apparently unaware.

I. E. Segal (New York, N. Y.).

Poncet, Jean. Sur les groupes simples localement compacts. I, II. C. R. Acad. Sci. Paris **238**, 192–194, 316–317 (1954).

Let G be a locally compact connected separable metric group possessing no non-trivial representations in a compact group. Let $H^*=G/H$ be a compact homogeneous space possessing an invariant measure under G. It is shown that there exist two elements in H^* the union of whose orbits under H is dense in H^*. It is shown further that every simple locally compact connected group G possesses a finitely generated dense subgroup. Remark: it follows from the recent work of Yamabe [Ann. of Math. (2) **58**, 48–54 (1953); these Rev. **14**, 948] that connected simple locally compact groups are Lie groups. It was shown by Auerbach and Ulam [C. R. Acad. Sci. Paris **201**, 117–119 (1935)] that every real or complex semisimple Lie group has a dense subgroup generated by four elements. *P. A. Smith.*

Vilenkin, N. Ya. On selection of a subgroup of elements of finite order as a direct summand in groups of type P. Mat. Sbornik N.S. **33(75)**, 37–44 (1953). (Russian)

Readers of earlier papers in this series [see these Rev. **13**, 110, 205, 206 for references] will recall that a group of type P is a locally compact abelian group G satisfying the second axiom of countability and associated with a fixed prime number p in this sense: for each g of G, if g is of finite order it is of order p^n for some n, and if g is of infinite order then $p^n g$ converges to the identity. If A denotes a subset of G, p^nA denotes the elements of A of height p^n in G and $p^\omega A$ denotes the intersection of the closures of all p^nA. The symbol ${}_n[A]$ denotes the elements of A of order p^n and the symbol ${}_\infty[A]$ denotes the union of all sets ${}_n[A]$: this is the set of all elements of A of finite order. Now the transfinite Ulm sequence, for as far as it goes,

$$G=G_0\supset G_1\supset G_2\supset\cdots\supset G_\alpha\supset\cdots$$

is defined by the conditions that $G_\alpha=p^\omega G_{\alpha-1}$, if α is a successor ordinal and $G_\alpha=\bigcap_{\beta<\alpha}G_\beta$, for all $\beta<\alpha$ if α is a limit ordinal. This study is restricted to groups of type P for which

$$(1)\qquad p^nG\cap G_\alpha=p^nG_\alpha,\quad\text{for every }\alpha.$$

The group G is said to satisfy a condition of "servantnost'" with respect to the subgroups G_α; perhaps "compliance" is an acceptable translation.

Let the closure of the subgroup ${}_\infty[G]$ be denoted by G^1. Let the factor group G/G^1 be denoted by G^*. This paper is restricted to groups for which G^* has no elements of finite order:

$$(2)\qquad {}_\infty[G^*]=0.$$

The author seeks conditions which will make G^1 a direct summand of G. The following conditions are listed as obviously necessary:

$$(3)\qquad \overline{p^nG_\alpha}\cap G^1=\overline{p^n(G^1)_\alpha}=\overline{{}_\infty[\overline{p^nG_\alpha}]}\equiv[\overline{p^nG_\alpha}]^1$$

$$(4)\qquad \overline{p^nG_\alpha}+G^1=\varphi_*^{-1}(\overline{p^nG_\alpha}^*),$$

where φ_* denotes the natural map of G onto G^*, and

$$(5)\qquad G^1\cap[p^nG+\overline{p^mG_\alpha}]=p^nG_1+\overline{p^m(G^1)_\alpha}.$$

Suppose that some compact and open subgroup H of G is marked out in G ("otmechni"), and that H is completely proper: as the reader is reminded in a later equation this means:

$$(10)\qquad p^mH\cap p^{m+n}G_\alpha=p^m(H\cap p^nG_\alpha).$$

Then if H, too, is decomposed into direct summands, the following relation will hold:

$$(6)\qquad H\cap[G^1+p^nG]=H\cap G^1+H\cap p^nG_\alpha.$$

It is assumed that the group G is fully properly stratified, so that these conditions also hold:

$$(7)\qquad p^nG\cap\overline{p^{n+m}G_\alpha}=p^n(\overline{p^mG_\alpha})$$

$$(8)\qquad {}_m[p^nG_\alpha]=\overline{{}_m[p^nG_\alpha]}$$

$$(9)\qquad G^1\cap G_\alpha=(G^1)_\alpha=(G_\alpha)^1.$$

We can now state the theorem to whose proof this note is devoted. Theorem. Let the group G of type P and the compact open subgroup H be given satisfying all the conditions enumerated above. Then G^1, defined above, is a direct summand of G and also induces a direct decomposition of H. *L. Zippin* (Flushing, N. Y.).

Matsushita, Shin-ichi. Über einen Satz von K. Iwasawa. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 59–61 (1953).

An incorrect proof of a result of Iwasawa [Proc. Imp. Acad. Tokyo **20**, 67–70 (1944); these Rev. **7**, 240]. *I. E. Segal* (New York, N. Y.).

Matsushita, Shin-ichi. Plancherel's theorem on general locally compact groups. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 63–70 (1953).

For any locally compact group G, the space $L_2(G)$ is mapped unitarily into the space of square-integrable functions on the space Ω of elementary positive definite functions on G, via $f\to F$, where $F(\omega)=\int\omega(x)f(x)dx$. The most novel part of this result is the existence of a relevant "random" measure on Ω, which would seem quite dubious. *I. E. Segal* (New York, N. Y.).

Sunouchi, Haruo. An extension of the Plancherel formula to unimodular groups. Tôhoku Math. J. (2) **4**, 216–230 (1952).

A "Plancherel theorem" for certain double representations of locally compact unimodular groups along the lines of Godement [J. Math. Pures Appl. (9) **30**, 1–110 (1951); these Rev. **13**, 12]. Godement treated the case where the associated rings of operators were of finite class. This paper does not assume those rings to be of finite class but assumes there is no purely infinite part and then proceeds essentially by breaking down the ring, through decompositions of the identity, into finite parts. To the reviewer it seems unjustified to call such theorems "Plancherel theorems" since they deal only with a decomposition into irreducible parts, which has long been recognized to be possible in many ways, and avoid the critical problem of classifying those irreducible parts. Hence they do not at all reveal the structure of the rings involved as the classical Plancherel theorem does for the abelian case. *W. Ambrose* (Cambridge, Mass.).

NUMBER THEORY

Moessner, Alfred. Einige zahlentheoretische Untersuchungen und Resultate. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 129–132 (1953). (Serbo-Croatian summary)

The author shows how an application of the identity

$$(a^2+b^2)^2=(a^2-b^2)^2+(2ab)^2$$

can produce solutions in integers of such systems as

$$2(A^{2n}-C^{2n})=B+D,\quad 2(A^{4n}-C^{4n})=B^2+D^2$$

and

$$2(P^n+Q^n)=x+y+z+w$$
$$2(P^{2n}+Q^{2n})=x^2+y^2+z^2+w^2$$
$$2(P^{3n}+Q^{3n})=x^3+y^3+z^3+w^3.$$

D. H. Lehmer (Berkeley, Calif.).

Scholomiti, N. C. An expression for the Euler ϕ-function. Amer. Math. Monthly 61, 36–37 (1954).

Buquet, A. Sur un critère d'indépendance de plusieurs solutions données de l'équation diophantienne en nombres rationnels $x^3+dx+e=z^2$. Mathesis 62, 281–289 (1953).

The writer continues his investigations [Mathesis 61, 181–193 (1952); these Rev. 14, 450] of geometric properties of the rational solutions of $x^3+dx+e=z^2$. Two rational points A_1 and A_2 on the curve determine a third such point called the resultant [cf. the earlier paper or the review thereof] designated by A_1A_2. This operation is commutative and associative, and so a set of points $A_1, A_2, \cdots, A_r$ determine a system $\sum n_iA_i$, the n_i being any integers. Systems of points are classified in various ways, the details being too numerous to permit delineation here. The classification enables the writer to formulate an elaborate algorithm which is employed to test the independence of solutions. *I. Niven* (Eugene, Ore.).

Ljunggren, Wilhelm. On an improvement of a theorem of T. Nagell concerning the Diophantine equation

$$Ax^3+By^3=C.$$

Math. Scand. 1, 297–309 (1953).

T. Nagell [J. Math. Pures Appl. (9) 4, 209–270 (1925)] proved that $Ax^3+By^3=C$ has at most one solution in integers if $C=1$ or 3, both $A>1$ and $B>1$ if $C=1$, and $(AB, 3)=1$ if $C=3$; exception must be made in the one case $2x^3+y^3=3$ with the two solutions (1, 1) and (4, −5), but this case is ignored in what follows. Any solution has the property that $C^{-1}(xA^{1/3}+yB^{1/3})=\xi^{2^r}$, where ξ is the fundamental unit in the field $R((AB^2)^{1/3})$. Nagell determined an upper limit for the non-negative rational integer r in certain cases, and P. Häggmark [Ark. Mat. 1, 279–294 (1950); these Rev. 12, 481] treated some others. In the present paper the best possible result $r\leqq 1$ is established for all cases. *I. Niven* (Eugene, Ore.).

Palamà, Giuseppe. Su di una questione relativa a somme uguali di potenze simili. Boll. Un. Mat. Ital. (3) 8, 286–293 (1953).

Denote by $M(n, r)$ the smallest value of p for which the diophantine equations

$$(1)\quad a_1^s+\cdots+a_p^s=a_p^s+\cdots+b_p^s \qquad (s=1, \cdots, n, n+2, n+4, \cdots, n+rn)$$

has solutions $a_1, \cdots, b_p$ in which the a's are not a permutation of the b's. Let $\alpha(k)$ denote the least value of p for which (1) has solutions for $s=k, k-2, \cdots, s\geqq 0$, so that $M(0, r)=\alpha(2r)$, and $M(1, r)=\alpha(2r+1)$. It is shown that

$$M(2, 2)=4,\quad M(2, 3)=5,\quad M(2, 4)\leqq 7,$$
$$M(2, 5)\leqq 15,\quad M(2, 6)\leqq 15,\quad M(2, 7)\leqq 24,\quad M(2, 8)\leqq 29.$$

It is conjectured that $M(2, r)=\alpha(2r+2)$. Furthermore

$$M(3, 2)=\alpha(7)=5,\quad M(3, 3)\leqq 7,\quad M(3, 4)\leqq 11,$$
$$M(3, 5)\leqq 17,\quad M(3, 6)\leqq 21,\quad M(3, 7)\leqq 30,\quad M(4, 2)\leqq 8,$$
$$M(4, 3)\leqq 11,\quad M(5, 2)\leqq 11.$$

D. H. Lehmer (Berkeley, Calif.).

Gloden, A. Un procédé de formation de systèmes multigrades normaux. Bull. Soc. Roy. Sci. Liège 22, 474–480 (1953).

Given two sets of n numbers having equal sums of like powers, say $\sum a_i^k=\sum b_i^k$ for $k=1, \cdots, n$, a procedure is sketched for extension to $n+1$, i.e., to obtain two sets of $n+1$ numbers having equal sums of kth powers for all positive integers $k\leqq n+1$. *I. Niven* (Eugene, Ore.).

Singh, Raghuraj. On multigrade equations of the third order. J. Sci. Res. Benaras Hindu Univ. 3, 1–4 (1953).

If a_1, a_2, a_3 and b_1, b_2, b_3 are two solutions of $x_1^2+x_2^2+x_3^2=n$, then

$$\sum_{i=1}^{3}\{(c+a_i)^j+(c-a_i)^j\}=\sum_{i=1}^{3}\{(c+b_i)^j+(c-b_i)^j\},\quad j=1, 2, 3.$$

This device clearly provides a class of solutions of the simultaneous system $\sum_{i=1}^{6}x_i^j=\sum_{i=1}^{6}y_i^j$ for $j=1, 2, 3$, but it is not clear that, as the author claims, all solutions are thus obtained. *I. Niven* (Eugene, Ore.).

Carlitz, L. A special congruence. Proc. Amer. Math. Soc. 4, 933–936 (1953).

The author proves the congruence

$$p+(p-1)\sum_{0<s(p-1)<m}\binom{m}{s(p-1)}\equiv 0 \pmod{p^{r+1}},$$

where p is a prime such that $p^r|m$ and $p\geqq 3$. For $r=0$ this result is due to Hermite. The author's proof was suggested by Nielsen's proof [Ann. Mat. Pura Appl. (3) 22, 249–261 (1914)] of Hermite's formula. *A. L. Whiteman.*

Kanold, Hans-Joachim. Untere Schranken für teilerfremde befreundete Zahlen. Arch. Math. 4, 399–401 (1953).

Let m_1 and m_2 be amicable integers (i.e.,

$$\sigma(m_1)=\sigma(m_2)=m_1+m_2$$

where $\sigma(m)$ is the sum of all divisors of m) which are relatively prime. It is proved that m_1m_2 has more than 20 prime factors, and that $m_1>10^{22}$, $m_2>10^{23}$. *I. Niven.*

Silverman, Louis L. Functional generalization of Bernoulli numbers. Riveon Lematematika 7, 33–37 (1954). (Hebrew. English summary)

In terms of an integer p and an arbitrary power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$, the author discusses coefficients B_n generated by

$$\left(\sum_{n=0}^{\infty}B_nx^n/n!\right)\left(1+\sum_{k=p}^{\infty}a_kx^{k+1-p}\right)=1.$$

These include the Bernoulli and Euler numbers and other familiar numbers when $y=f(x)$ is an entire function satisfying $y^{(n)}=y$ $(n=1, 2, 4)$. The case of $n=3$ is touched upon.
D. H. Lehmer (Berkeley, Calif.).

Jarden, Dov. On the algebraic factors of the numbers V_{5n}/V_n (n odd) in the sequence associated with Fibonacci's sequence. Riveon Lematematika 7, 23–25 (1954). (Hebrew. English summary)

Proof is given of the following theorem, stated without proof by M. Kraitchik [Recherches sur la théorie des nombres, t. I, Gauthier-Villars, Paris, 1924, p. 81]: Let $A_n=3+V_{2n}-5U_n$, $B_n=3+V_{2n}+5U_n$ where U_n are the Fibonacci numbers and $V_n=U_{2n}/U_n$ the associated Fibonacci numbers. Then, for n odd, $A_n|A_{mn}$ for $m\equiv\pm1$ (mod 10) and $A_n|B_{mn}$ for $m\equiv\pm3$ (mod 10). The same results hold if the letters A and B are interchanged. (Paraphrased from author's summary.) *E. G. Straus.*

Subba Rao, K. Some properties of Fibonacci numbers. Amer. Math. Monthly 60, 680–684 (1953).

Denote by U_n the nth Fibonacci number

$$(U_0=U_1=1,\ U_n=U_{n-1}+U_{n-2}).$$

The author proves by induction several new identities, e.g.,

$$\sum_{k=0}^{n-1} U_kU_{k+2}U_{k+4}=\tfrac{1}{10}[U_{3n+7}+(-1)^{n+1}16U_n-5].$$

Further he proves the following theorem: Let $k>1$ be an integer. Then there exists an $l<k$ so that if $n_i>n_0=n_0(k)$, $1\leqq i\leqq k$, we have

$$U_{n_1+n_2+\cdots+n_k-1}<U_{n_1}\cdot U_{n_2}\cdots U_{n_k}<U_{n_1+n_2+\cdots+n_k-l+1}.$$

Also for $n>n_0$ there are exactly mk Fibonacci numbers between U_n^k and U_{n+m}^k. *P. Erdös* (South Bend, Ind.).

Lekkerkerker, C. G. Representation of natural numbers as a sum of Fibonacci numbers. Simon Stevin 29, 190–195 (1952). (Dutch)

Let $u_1=1$, $u_2=2$, and $u_n=u_{n-1}+u_{n-2}$ for $n\geqq3$. It is shown that every positive integer N has a unique representation in the form $N=\sum_{\nu=1}^k u_{i_\nu}$ with $i_{\nu+1}\geqq i_\nu+2$ for $\nu=1, 2, \cdots, k-1$, and that in this representation, $u_{i_k}\leqq N<u_{i_k+1}$. Moreover, the average number of summands required to represent the integers N such that $u_r\leqq N<u_{r+1}$ is asymptotic to cr as $r\to\infty$, where $c=\frac{1}{2}(1-5^{-\frac{1}{2}})$. *W. J. LeVeque.*

Eljoseph, Nathan. On the representation of a number as a sum of squares. Riveon Lematematika 7, 38–43 (1954). (Hebrew. English summary)

In the first part of this paper the author points out an error in the proofs of Theorems 4 and 10 of I. Niven's paper, Trans. Amer. Math. Soc. 48, 405–417 (1940) [these Rev. 2, 147]. He points out that instead of proving that every number $a+2b\sqrt{-m}$ (a, b, m rational integers, $m\geqq2$, square free) is the sum of squares of three integers in $R(\sqrt{-m})$, all that Niven's argument for Theorem 4 proves is: Either $a+2b\sqrt{-m}=\alpha^2+\beta^2+\gamma^2$ or $a+2b\sqrt{-m}=\alpha^2-\beta^2-\gamma^2$ where α, β, γ are integers in $R(\sqrt{-m})$. The author gives a valid counterexample to Theorem 10 and an invalid counterexample to Theorem 4. However Theorem 4 was disproved by C. L. Siegel [Ann. of Math. (2) 46, 313–339 (1945); these Rev. 7, 49], who gives an infinite number of imaginary quadratic fields, including $R(\sqrt{-7})$, in which 7 is not the sum of three squares.

In the second part the author investigates the possibility of expressing positive rational integers as sums of squares of numbers $a+b\sqrt{D}$ (a, b, D integers, $D\neq$square). He proves: Every positive rational integer n is the sum of (1) two squares, if and only if $D=-1$; (2) three squares, if $D=2, 3, 5, 6$ but not if $D\geqq8$ ($n=7$ is the common counterexample. This leaves the case $D=7$ open).
E. G. Straus (Los Angeles, Calif.).

***de Beaumont, Henry du Boscq. Démonstration du dernier théorème de Fermat.** Editions Industrielles, Techniques et Littéraires, Paris, 1953. ii+18 pp. (unnumbered)

The author begins his proof by introducing the birational transformation $(u-v^2)x=v$, and $(u+v^2)y=v$, where $x^n+y^n=1$. Unfortunately there is a rather serious nonsequitur in the paragraph common to pages 6 and 7 which the reviewer has been unable to rectify.
D. H. Lehmer (Berkeley, Calif.).

Inkeri, K. Abschätzungen für eventuelle Lösungen der Gleichung im Fermatschen Problem. Ann. Univ. Turkuensis. Ser. A. 16, no. 1, 9 pp. (1953).

This paper improves upon results recently obtained by Duparc and van Wijngaarden [Nieuw Arch. Wiskunde (3) 1, 123–128 (1953); these Rev. 15, 200] for lower bounds for x and z in Fermat's equation

$$x^l+y^l=z^l\ (x, y, z)=1,\quad 0<x<y<z.$$

It is shown that in Case I (xyz not divisible by l)

$$x>\{(2l^3+1)/\log(3l)\}^l,$$

while in Case II

$$x>l^{3l-4},\quad z>\tfrac{1}{2}l^{3l-1}.$$

These inequalities lead to inconceivably large numbers when the known lower limits for l in the two cases are substituted. Incidentally in Case II, Vandiver's old lower limit 619 may now be replaced by 2521. *D. H. Lehmer.*

HAVE

***Trost, Ernst. Primzahlen.** Verlag Birkhäuser, Basel-Stuttgart, 1953. 95 pp. 13.50 Swiss francs.

The purpose of this very concise book is to give a self-contained presentation of the problems and results of the theory of prime numbers which lie within the scope of the so-called elementary methods. In the opinion of this reviewer the book achieves this goal quite successfully. Starting from the unique factorization theorem, congruences, quadratic reciprocity, characters, and other preparatory material, the subject matter ranges from a discussion of criteria for primality through the prime-number theorem. Also treated in the course of the development are the representation of primes by quadratic forms, the method of Tchebychef for estimating $\pi(x)$, the sieve method of Brun, the representation of integers as sums of primes, and Dirichlet's theorem concerning the infinitude of primes in arithmetic progressions. In general, complete proofs are given. *H. N. Shapiro* (New York, N. Y.).

Singh, Daljit. Concerning the reciprocal of a prime. Amer. Math. Monthly 61, 32–34 (1954).

Sathe, L. G. On a problem of Hardy on the distribution of integers having a given number of prime factors. II. J. Indian Math. Soc. (N.S.) 17, 83–141 (1953).

The author completes the proof of the results announced in the first part of this paper [same vol., 63–82 (1953); these Rev. 15, 103]. *P. Erdös* (South Bend, Ind.).

Gel'fond, A. O. On an elementary approach to some problems from the field of distribution of prime numbers. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8, 21–26 (1953). (Russian)

Let χ be a real non-principal character modulo m and $q=\max_{k\geq 1}|\sum_{n=1}^{k}\chi(n)|$. The author gives an elementary proof that $|L(1,\chi)|>c/q\log q$ if $q>q_0$. The proof is based on the identity

$$\sum_{n=1}^{\infty}\chi(n)x^n/(1-x^n)=\sum_{n=1}^{\infty}x^n\prod_{p^r\|n}\{1+\chi(p)+\cdots+\chi^r(p)\}.$$

A lower bound is obtained for the right side by dropping all terms except those for which n is a square; the product is then at least 1. The left side is estimated by a number of partial summations and the identity

$$\frac{\chi(n)}{1-x^n}=\frac{1}{1-x}\frac{\chi(n)}{n}+\frac{1}{1-x}\chi(n)\left\{\frac{1-x}{1-x^n}-\frac{1}{n}\right\}.$$

By taking $x=1-c'/(q\log q)^2$, the result is obtained. When Pólya's result that $q\leq m^{\frac{1}{2}}\log m$ is used the author's result yields the estimate $|L(1,\chi)|>c''/m^{\frac{1}{2}}\log^2 m$ which is inferior to both Siegel's lower bound of $c(\epsilon)/m^{\epsilon}$ and the earlier bound of $C/m^{\frac{1}{2}}$.

A second result, proved in a similar manner, is that if $\Theta(n)$ is a completely multiplicative function taking only the values $0, 1, -1$ and if $\limsup_{n\to\infty} n^{-\frac{1}{2}}|\sum_{k=1}^{n}\Theta(k)|<1/12$, then $\sum_{n=1}^{\infty}\Theta(n)/n\neq 0$. By taking $\Theta(k)$ to be Liouville's function $\lambda(k)$, the author shows that the exponent $\frac{1}{2}$ cannot be increased, if the Riemann hypothesis holds.

L. Schoenfeld (Urbana, Ill.).

Tatuzawa, Tikao. On the product of $L(1,\chi)$. Nagoya Math. J. 5, 105–111 (1953).

Let k be a positive integer, let $\omega(k)$ be the number of distinct prime factors of k and let λ_k be the product of $L(1,\chi)$ extended over all characters χ modulo k with the exception of χ_0. Then for each $\epsilon>0$

$$c(\epsilon)k^{-\epsilon}<\lambda_k<\exp[c\{\log\log k+\omega(k)\}].$$

A similar result is obtained when the product λ_k is restricted to the characters for which $\chi(-1)=-1$. From the first result the author obtains upper and lower bounds for the product of the regulator and the class number of the cyclotomic field $P(e^{2\pi i/p})$ for an odd prime p; this result is sharper than a general result of R. Brauer [Amer. J. Math. 72, 739–746 (1950); these Rev. 12, 482]. From the second result, the author obtains upper and lower bounds for the first factor of the class number of $P(e^{2\pi i/p})$ these bounds being sharper than those of Ankeny and S. Chowla [Proc. Nat. Acad. Sci. U. S. A. 35, 529–532 (1949); these Rev. 11, 230].

L. Schoenfeld (Urbana, Ill.).

Turán, Paul. On an application of the typical means in the theory of the zeta-function of Riemann. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 239–251 (1952).

The following result is proved under the assumptions that $\frac{1}{2}\leq\alpha<1$ and $0<\epsilon<(1-\alpha)/10$. There exists a sequence $s_n=\sigma_n+it_n$ such that $t_n\to\infty$ as $n\to\infty$, $\alpha+\epsilon\leq\sigma_n<1$ and $|\zeta(s_n)|\leq\exp\{-(\log t_n)^{\frac{1}{2}(1-\alpha-4\epsilon)}\}$. Since the result is clearly trivial if there are an infinity of zeros in $\Re(s)\geq\alpha+\epsilon$, only the contrary case is considered; and for this case the author shows that the above is true with all $\sigma_n=\alpha+2\epsilon$.

The point of departure is the following modified form of the Perron-Hadamard formula, where $f(w)=\sum_{n=1}^{\infty}a_ne^{-\lambda_nw}$,

$$\sum_{n\leq N}a_n(\lambda_N-\lambda_n)e^{-\lambda_ns}$$
$$=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\left\{\frac{2}{w-s}\sinh\left(\frac{w-s}{2}\lambda_n\right)\right\}^2f(w)dw.$$

The author applies this result to $f(w)=\log 1/\zeta(w)$, $\lambda_n=\log n$ and $s=\alpha+2\epsilon+it$. The line of integration is moved to the left so that for $|\Im(w)|\geq\frac{1}{2}t$ the path is along $\Re(w)=\alpha+2\epsilon$; the remainder of the new integration path is along $\Re(w)=1$ except for a small detour to the right of $w=1$ and two horizontal paths. The real parts of both sides are taken and the integrals along all paths, except that part of $\Re(w)=\alpha+2\epsilon$ in the upper half-plane, are easily estimated. At this stage the result is

$$-\sum_{p^k\leq N}\frac{\cos(kt\log p)}{kp^{k(\alpha+2\epsilon)}}\log\frac{N}{p^k}$$
$$\leq\frac{1}{2\pi}\int_{\frac{1}{2}t}^{\infty}\Re f(\alpha+2\epsilon+iv)\left\{\frac{2}{v-t}\sin\left(\frac{v-t}{2}\log N\right)\right\}^2dv$$
$$+c_1(\epsilon)N^{1-\alpha-2\epsilon}\log\log N\cdot(\log t)/t.$$

That part of the integral having $v\geq 4t$ is easily estimated and is absorbed in the other error term. If $L=\max_{t/2\leq v\leq 4t}\log 1/|\zeta(\alpha+2\epsilon+iv)|$ the rest of the integral has the estimate

$$\frac{1}{2\pi}L\int_{\frac{1}{2}t}^{4t}\left\{\frac{2}{v-t}\sin\left(\frac{v-t}{2}\log N\right)\right\}^2dv\leq c_2L\log N,$$

and the sum of all the terms on the left for which $k\geq 2$ has the easily obtained estimate $c_3(\epsilon)\log N$. (Correction: both sums in (6.1) of the paper should be restricted by the condition $k\geq 2$.) Consequently, after division by $\log N$ one obtains

$$-\sum_{p\leq N}\frac{\cos(t\log p)}{p^{\alpha+2\epsilon}}\left(1-\frac{\log p}{\log N}\right)\leq c_2L$$
$$+c_4(\epsilon)\{1+N^{1-\alpha-2\epsilon}(\log t)/t\}.$$

At this point the author extends a theorem of Bohr and Landau used for a similar purpose in the half plane $\Re(s)>1$. The new result is that to each $N>c_5$ there corresponds a τ_N such that

$$\exp(2N^3)\leq\tau_N\leq 2\exp(2N^3)\quad\text{and}\quad\cos(\tau_N\log p)\leq-\tfrac{1}{2}$$

for all primes $p\leq N$. Taking $t=\tau_N$ in the above result, the left side is greater than or equal to

$$\tfrac{1}{2}\sum_{p\leq\frac{1}{2}N}p^{-\alpha-2\epsilon}\left(1-\frac{\log p}{\log N}\right)\geq\frac{1}{2}\left(1-\frac{\log\frac{1}{2}N}{\log N}\right)\sum_{p\leq\frac{1}{2}N}p^{-\alpha-2\epsilon}$$
$$\geq c_6(\epsilon)N^{1-\alpha-2\epsilon}/\log^2 N.$$

The proof is now easily completed. *L. Schoenfeld.*

Koksma, J. F., and Lekkerkerker, C. G. A mean-value theorem for $\zeta(s,w)$. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 446–452 (1952).

For $\Re(s)>1$ and $0\leq w\leq 1$ let

$$\zeta^*(s,w)=\sum_{n=1}^{\infty}(n+w)^{-s}=\zeta(s,w)-w^{-s}$$

and let the function be defined for other values of s by analytic continuation. The authors give a straightforward

proof of the following results for the mean value, taken with respect to w,

$$f(s)=\int_0^1|\zeta^*(s,w)|^2dw.$$

If $|t|\geqq 3$ then $f(\frac{1}{2}+it)<64\log|t|$. Their second result may be phrased as follows: if $|t|\geqq 3$ and $\frac{1}{2}<\sigma\leqq 1$ then

$$|f(\sigma+it)-(2\sigma-1)^{-1}|\leqq|t|^{1-2\sigma}(32\log|t|+(2\sigma-1)^{-1});$$

hence, if $\frac{1}{2}+(2A\log|t|)^{-1}\leqq\sigma\leqq 1$ and $A\geqq 32$, then the right side may be replaced by $2A|t|^{1-2\sigma}\log|t|$. The proof is carried out with the aid of a weak form of the approximate functional equation for $\zeta^*(s,w)$; for a stronger form of such an equation, see Čudakov [Ann. of Math. (2) **48**, 515–545 (1947); these Rev. **9**, 11]. *L. Schoenfeld* (Urbana, Ill.).

van Wijngaarden, A. On the coefficients of the modular invariant $J(\tau)$. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 389–400 (1953).

The author presents a table of the coefficients $c(n)$ of Klein's absolute modular invariant $J(\tau)=\sum_{n=-1}^{\infty}c(n)x^n$, where $x=\exp(2\pi i\tau)$, and $\Re\tau>0$ for $n\leqq 100$. Only the values up to $n=25$ were previously available. The $c(n)$, which are positive rational integers, grow very rapidly with n, like $\exp c\sqrt{n}$, where c is a certain positive constant. $c(100)$ is a number of 53 digits.

The calculation was based on the formula

$$J(\tau)=\frac{2}{27}(\theta_2{}^8+\theta_3{}^8+\theta_4{}^8)(\theta_2{}^{-8}+\theta_3{}^{-8}+\theta_4{}^{-8})$$

appearing in van der Pol's paper [same Proc. **54** = Indagationes Math. **13**, 261–271, 272–284 (1951); these Rev. **13**, 135]. Checks on the $c(n)$ were made on the basis of congruences developed by D. H. Lehmer [Amer. J. Math. **64**, 488–502 (1942); these Rev. **3**, 272], the reviewer [ibid. **71**, 136–148, 373–386 (1949); these Rev. **10**, 357], and the author. The author's congruences connect $c(n)$ with certain restricted partition functions tabulated by Watson [Proc. London Math. Soc. (2) **42**, 550–556 (1937)].

Wijngaarden's table verifies a conjecture of the reviewer made in one of the papers cited above for the new values of n, and also for $n=128$; for this purpose the author calculated $c(128)$ modulo 2^{30} from one of Lehmer's formulas.
J. Lehner (Los Alamos, N. Mex.).

Meinardus, Günter. Über das Partitionenproblem eines reell-quadratischen Zahlkörpers. Math. Ann. **126**, 343–361 (1953).

For any totally positive integer μ of the real quadratic field with discriminant d, let $P(\mu)$ be the number of representations of μ as a sum of totally positive integers of that field, the order of the terms in any representation being irrelevant. The author proves that

$$\log P(\mu)=3d^{-1/6}\{\zeta(3)N\}^{1/3}+\alpha_1\log^2 N+\alpha_2\log N+\delta+O(N^{-1/21})$$

as $N\to\infty$, where N is the norm of μ, ζ is the Riemann zeta function, α_1 and α_2 are independent of μ, and δ is a bounded function of μ, whose exact value is given in terms of infinite series. *T. Estermann* (London).

Urazbaev, B. M. On the discriminant of a cyclic field of prime degree. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 19–32 (1950). (Russian)

Let K be a cyclic extension of the field of rationals of odd prime degree p. Then the discriminant D of K has the form $D=p^a\prod q_i{}^{p-1}$, where the q_i are distinct rational prime numbers of the form $np+1$ and $a=0$ or $a=2(p-1)$.
W. H. Mills (New Haven, Conn.).

Urazbaev, B. M. On the number of cyclic fields of prime degree with given discriminant. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 53–67 (1951). (Russian)

Let p be an odd prime, and let $q_1, q_2, \cdots, q_k$ be distinct primes of the form $np+1$. Then the number of cyclic extensions of the field of rationals of degree p and discriminant $(p^a q_1\cdots q_k)^{p-1}$ is $(p-1)^k$ if $a=2$ and is $(p-1)^{k-1}$ if $a=0$. There are no other possible discriminants for cyclic extensions of the rationals of degree p. [Cf. the paper reviewed above.] *W. H. Mills* (New Haven, Conn.).

Urazbaev, B. M. On indexes of algebraic equations. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 33–41 (1950). (Russian)

Let R be the field of rational numbers, and α a root of the irreducible polynomial $f(x)$ with integral coefficients and leading coefficient 1. Then the index m of $f(x)$ is defined by $d(\alpha)=Dm^2$, where $d(\alpha)$ is the discriminant of $f(x)$ and D the discriminant of the field $R(\alpha)$ over R. Let p be a rational prime, and let $f(x)\equiv\prod p_i(x)^{e_i}$ be the factorization of $f(x)$ modulo p. Put $f(x)=\prod p_i(x)^{e_i}+pM(x)$. Then p is a factor of m if and only if there exists a j such that $e_j\geqq 2$ and $p_j(x)$ divides $M(x)$ modulo p. *W. H. Mills*.

Urazbaev, B. M. On the density of distribution of points of cyclic fields of prime degree. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 25–36 (1951). (Russian)

Let p be an odd prime. If α is an element of K, a cyclic extension of the field of rational numbers of degree p, then α can be regarded as the point $(\alpha^{(1)}, \cdots, \alpha^{(p)})$ of p-dimensional Euclidean space, where $\alpha^{(1)}, \cdots, \alpha^{(p)}$ are the conjugates of α. The author considers the set of all cyclic extensions of the rationals of degree p, and the set of all algebraic integers of these fields that correspond to points in the interior of a sphere of radius R. The number M_p of such points is given by

$$M_p=\sigma_pR^p+O(R^{p-1})\text{ if }p>3,\quad M_3=\sigma_3R^3\log R+O(R^3),$$

where the σ_p are non-zero constants that depend only on p.
W. H. Mills (New Haven, Conn.).

Urazbaev, B. M. On the density of distribution of cyclic fields of prime degree. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 37–52 (1951). (Russian)

The number L of cyclic extensions of the field of rationals having degree p and discriminant not greater than N is given by the asymptotic formula $L=CN+O(N^{1-1/(p-1)})$, where C is a positive constant depending only on p.
W. H. Mills (New Haven, Conn.).

Krubeck, Eleonore. Über Zerfällungen in paarweis ungleiche Polynomwerte. Math. Z. **59**, 255–257 (1953).

The author proves the following result. To every polynomial

$$f(x)=a_0+a_1x+\cdots+a_nx^n\ (a_n>0)$$

there exists a positive integer s such that every integer $N\geqq 0$ is represented by $N=r+f(x_1)+f(x_2)+\cdots$, where $0\leqq r<s$, and where x_i are positive integers such that $0<x_1<x_2<\cdots$, and $0<f(x_1)<f(x_2)<\cdots$. The method of

proof is based upon that of R. Sprague [see Math. Z. 51, 289–290, 466–468 (1948); these Rev. 10, 283, 514; also H. Richert, Norsk Mat. Tidsskr. 31, 120–122 (1949); these Rev. 11, 646] and depends on a solution of the Tarry-Escott problem. *D. H. Lehmer* (Berkeley, Calif.).

Rumney, Max. Equations in polynomials. Math. Gaz. 37, 261–264 (1953).

The paper contains a brief discussion of some properties of polynomials in a single indeterminate with rational coefficients. In particular the solution of the equation $AU+BV=k$ by continued fractions is indicated; the case in which the coefficients are integral is also mentioned. As an application the following result is obtained. If $U_1, V_1; \cdots; U_{n+1}, V_{n+1}$ denote solutions of $AU+BV=P$ and $H_i(u, v)$ are homogeneous polynomials, then one has, for the determinant of order $n+1$, $|H_i(U_j, V_j)|\equiv 0 \pmod{P^{\frac{1}{2}n(n+1)}}$. *L. Carlitz* (Durham, N. C.).

Carlitz, L. The Schur derivative of a polynomial. Proc. Glasgow Math. Assoc. 1, 159–163 (1953).

Let $f(x)=f(x_1, \cdots, x_k)$ denote a polynomial in k indeterminates with integral coefficients. For $p\neq 0$ define the Schur derivatives of $f(x)$ by

$$f_m'(x)=(f^{p^{m+1}}(x)-f^{p^m}(x^p))/p^{m+1},$$
$$f_m^{(r+1)}(x)=(f_{m+1}^{(r)}(x)f^{p^{m+1}}(x)-f_m^{(r)}(x)f^{p^{m+r}}(x^p))/p^{m+1},$$

with $r\geqq 1$. Generalizing the results of Schur [S.-B. Preuss. Akad. Wiss. 1933, 145–151] the author shows that if p is a prime, then $f_m^{(r)}(x)$ has integral coefficients for $1\leqq r\leqq p-1$. More precisely, he determines the residue of $f_m^{(r)}(x)$ (mod p^m). Finally, he considers generalizations of his results which are valid for any commutative ring that contains the rational integers. *A. L. Whiteman.*

Carlitz, L. Pairs of quadratic equations in a finite field. Amer. J. Math. 76, 137–154 (1954).

Let $Q_1, \cdots, Q_r$ denote r nonsingular quadratic forms with coefficients in $GF(q)$ with nonoverlapping sets of unknowns; let $\beta_1, \cdots, \beta_r$ be distinct and α, β arbitrary numbers of $GF(q)$. Let $N(\alpha, \beta)$ denote the number of solutions of the system

$$(*)\qquad \begin{aligned} Q_1(\xi^{(1)})+\cdots+Q_r(\xi^{(r)})&=\alpha,\\ \beta_1Q_1(\xi^{(1)})+\cdots+\beta_rQ_r(\xi^{(r)})&=\beta, \end{aligned}$$

where $\xi^{(i)}$ stands for $(\xi_1^{(i)}, \cdots, \xi_{t_i}^{(i)})$. The author first obtains explicit results for $N(\alpha, \beta)$ when the Q's satisfy certain additional hypotheses. In some cases $N(\alpha, \beta)$ is exhibited in terms of Jacobsthal sums. Next the author discusses a conjecture of Weil [Bull. Amer. Math. Soc. 55, 497–508 (1949); these Rev. 10, 592] for the homogeneous system (*) with $\alpha=\beta=0$. Finally he investigates briefly the case $q=2^n$. *A. L. Whiteman* (Princeton, N. J.).

Carlitz, L. The class number of an imaginary quadratic field. Comment. Math. Helv. 27 (1953), 338–345 (1954).

Let $h(d)$ denote the class number of the imaginary quadratic field $R(\sqrt{d})$ of discriminant d. Let p be an odd prime divisor of d and let $n\geqq 0$. The author proves that for $d<-4$,

$$h(d)\equiv -2c\left(\frac{q}{p}\right)\sum_{1\leqq s<q/2}\left(\frac{q_0}{s}\right)B_k\left(\frac{s}{q}\right) \pmod{p^{n+1}},$$

where $d=(-1)^{(p-1)/2}pq_0$, $q=|q_0|$, (q_0/s) is the Kronecker symbol, $B_k(x)$ is the Bernoulli polynomial of degree $k=\frac{1}{2}(p-1)p^n+1$ and $c=1+\frac{1}{2}p^n$ for $n\geqq 1$, while $c=2$ for $n=0$. *A. L. Whiteman* (Princeton, N. J.).

Mordell, L. J. On the linear independence of algebraic numbers. Pacific J. Math. 3, 625–630 (1953).

Let K be an algebraic number field and $x_1, \cdots, x_s$ roots of the equations $x_i^{n_i}=a_i$ $(i=1, 2, \cdots, s)$ and suppose that (1) K and all x_i are real, or (2) K includes all the n_ith roots of unity, i.e. $K(x_i)$ is a Kummer field. The following theorem is proved. A polynomial $P(x_1, \cdots, x_s)$ with coefficients in K and of degrees in x_i, less than n_i for $i=1, 2, \cdots, s$, can vanish only if all its coefficients vanish, provided that the algebraic number field K is such that there exists no relation of the form $x_1^{\nu_1}x_2^{\nu_2}\cdots x_s^{\nu_s}=a$, where a is a number in K unless $\nu_i\equiv 0 \bmod n_i$ $(i=1, 2, \cdots, s)$. When K is of the second type, the theorem was proved earlier by Hasse [Klassenkörpertheorie, Marburg, 1933, pp. 187–195] by help of Galois groups. When K is of the first type and K also the rational number field and the a_i integers, the theorem was proved by Besicovitch in an elementary way. The author here uses a proof analogous to that used by Besicovitch [J. London Math. Soc. 15, 3–6 (1940); these Rev. 2, 33]. *H. Bergström* (Göteborg).

→*Furtwängler, Philipp. Allgemeine Theorie der algebraischen Zahlen.** Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 2, 19. Band I. Algebra und Zahlentheorie. 2. Teil. C. Reine Zahlentheorie. Heft 8, Teil II. B. G. Teubner Verlagsgesellschaft, Leipzig, 1953. 50 pp. DM 6.00.

This article of the new edition of the Enzyklopädie der mathematischen Wissenschaften is from a manuscript of Furtwängler written by 1938 and later largely completed by Eichler, Hasse and Jehne. It covers that part of algebraic number theory which is concerned with properties of finite algebraic number fields in general; class field theory, the theory of complex multiplications and other special theories on algebraic number fields are to be given in other articles. The material is well prepared and neatly arranged in a relatively small space. The main contents are as follows. 1. The ideal theory of algebraic number fields; methods of Dedekind and Kronecker. 2. Kummer-Hensel's method with valuation theory. 3. The structure of residue-class rings modulo integral ideals. 4. The ramification theory. 5. Ideal classes and units. 6. Decomposable forms and Klein's lattices. 7. Orders and their conductors. 8. Artin's conductors and Artin's L-functions. 9. Calculation of class numbers by analytic method. *K. Iwasawa* (Cambridge, Mass.).

Hasse, Helmut. Gausssche Summen zu Normalkörpern über endlich-algebraischen Zahlkörpern. Abh. Deutsch. Akad. Wiss. Berlin, Kl. Math. Nat. 1952, no. 1, 19 pp. (1952).

Suppose that N is a normal extension with the Galois group G whose characters are denoted by χ, over the algebraic number field K. The author presents a detailed sketch with definitions for problems and conjectures related to the determination and interpretation of the factors occurring in the functional equations of Artin's L-series [Artin, Abh. Math. Sem. Hamburg. Univ. 8, 292–306 (1930); J. Reine Angew. Math. 164, 1–11 (1931)]. In order to make the connection with Hecke's abelian L-series the Γ-factors occurring in the functional equation are used to define infinite contributions to modified L-series $\bar{L}(s|\chi, N/K)=\prod_p L_p(s|\chi, N/K)$ where $L_p(\cdots)$ is defined for finite prime divisors p of K by the customary sum for its logarithm, and for complex infinite p of K, for example, $L_p(\cdots)=\gamma(s)^r\gamma(s+1)^r$ with $r=\chi(1)$ and $\gamma(s)=\frac{1}{2}\pi^{-s/2}\Gamma(s/2)$ is taken. Similarly modified conductors $\bar{f}(\chi, N/K)$, differents

and discriminants are defined as products of local conductors, etc. using appropriately defined Hilbert groups for the infinite prime divisors. The behaviour of these extended series and divisors as functions of the character χ, such as reduction to a factor group and induced characters, are stated, especially how they reduce to abelian L-series and conductors so that the new definitions appear as natural generalizations. Now let $N(K, \chi)$ denote the finite part of the absolute norm $N_K(\bar{F})$ of the extended conductor $\bar{D}(K)^r\tilde{f}(\chi, N/K)=\bar{F}$ where $\bar{D}(K)$ stands for the absolute different of K and where $r=\chi(1)$; the author sets $M(s, \chi)=N(K, \chi)^{s/2}\bar{L}(s|\chi, N/K)$ so that

$$\tau_K(\chi)=[N_K(\tilde{f}(\chi, N/K))]^{\frac{1}{2}}[M(s, \chi)M(1-s, \bar{\chi})^{-1}]$$

is for abelian N/K a gaussian sum of Hecke. At this juncture (global properties like the form of a functional equation do not necessarily reduce to formally equivalent local properties) the author applies R. Brauer's theorem [Ann. of Math. (2) 48, 502–514 (1947); these Rev. 8, 503] which states that each character χ of G can be expressed in at least one manner as a sum $\sum_\psi g(\psi)\psi^*$ with integer coefficients $g(\psi)=g$ of extended characters ψ^* of G belonging to abelian characters ψ [$\psi(1)=1$] of subgroups H of G. The formal group-theoretical properties of the L-series and conductors yield then that the formally defined gaussian sum $\tau(\chi, N/K)=\prod_\psi T(\Lambda, \psi)^g[N_N(\bar{D}(K))^r]^{-\frac{1}{2}}$ with $r=\chi(1)$, where $T(\Lambda, \psi)$ stands for the extended abelian gaussian sum $\tau(\psi, \Lambda/K)[N_\Lambda(\bar{D}(\Lambda))^\rho]^{\frac{1}{2}}$ with $\rho=\psi(1)$, of the extension Λ/K belonging to H the proper invariantive behaviour in the functional equation. Formal definitions like the last yield directly from the known abelian case relations like $|\tau(\chi, N/K)|=N_K(f(\chi, N/K))=N_0$.

One of the principal problems mentioned by the author concerns the arithmetic nature of these generalized gaussian sums $\tau(\chi, N/K)$; for example, it is conjectured that they are algebraic integers and lie in the field of N_0th roots of unity over the field of the character χ; also, their ideal-theoretic factorization is an open problem. Next the author turns to the abelian case in order to formulate a potential mode of procedure for the solution of these important problems. Brauer's reduction to extended abelian characters permits one to apply methods of localization which are familiar in class field theory [a character ψ is the product of its local components ψ_p which belong to the right local conductors] and which lead (i) to p-adic gaussian sums $T_p(\psi_p)$ which can be defined for the components ψ_p by strictly local methods, and then (ii) to gaussian sums in finite fields according to general (as yet unavailable to this reviewer) results of E. Lamprecht in which case connections with earlier results of Davenport and the author concerning function fields of one variable [J. Reine Angew. Math. 172, 151–182 (1934)] may be brought to bear upon these problems. Thus the p-component of a general (not necessarily abelian) gaussian sum $T(\chi, N/K)=\prod_\psi T(\Lambda, \psi)^g$ is defined as $T_p(\chi, N/K)=\prod_\psi\prod_q T_q(\psi_q)^g$ where q runs over all prime divisors of Λ which divide p. It is to be noted that these p-components of $T(\chi, N/K)$ depend upon the representation of χ as $\sum_\psi g(\psi)\psi^*$ which is not necessarily unique, and the author indicates the existence of examples for which $\prod_\psi T_p(\psi)^g=T_p$ is not equal to 1 for every relation $\sum_\psi g(\psi)\psi^*=0$. In this connection the author formulates several conjectures (with strong supporting indications related to his paper with Davenport, loc. cit.) for the products T_p, for example that they are powers of i also for the finite prime divisors. The author closes this interesting announcement with further significant conjectures related to global product formulas for abelian gaussian sums and their connection with similar theorems for product relations involving the zeros of L-series which belong to function fields. Detailed proofs for parts of the author's sweeping indications are promised for subsequent publications.

O. F. G. Schilling (Chicago, Ill.).

Schilling, O. F. G. Necessary conditions for local class field theory. (Remarks to a paper of M. Moriya.) Math. J. Okayama Univ. 3, 5–10 (1953).

Let F be a field which is complete for a discrete valuation of rank one. Let F' denote the residue class field of F for this valuation. Call F' adequate if it is perfect and if, for every positive integer n, the algebraic closure of F' contains exactly one extension of F' which is of degree n. It is known from earlier work of Moriya and Nakayama [see these Rev. 7, 363] that if F' is adequate then local class field theory holds over F. Conversely, it was shown more recently by Moriya [same J. 2, 13–20 (1952); these Rev. 14, 452] that if the (reciprocity-) isomorphism theorem and the limitation theorem of the local class field theory hold over F then F' is adequate. It is shown here that the following conditions (A) and (B) (taken jointly), which would also follow from local class field theory over F, imply also that F' is adequate. (A) For each prime power p^m, there exists a cyclic extension Z/F of degree p^m whose norm class group contains an element of order p^m. (B) If k/F is a finite algebraic extension then the reciprocity isomorphism holds for cyclic unramified extensions K/k of prime degree p and for abelian extensions K/k of type (p, p). *G. Hochschild.*

Lamprecht, Erich. Über s-Differenten und Differentiale algebraischer Funktionenkörper einer Veränderlichen. Arch. Math. 4, 412–424 (1953).

Instead of the trace for a finite separable extension K of a field k the author considers a linear function s defined on K with values in k for a general (non-separable) extension K/k. Thus he defines the s-different of K/k using this function s instead of trace in Dedekind's definition of different. He considers first the case of discretely valued perfect fields and then the global case of algebraic function fields of one variable. He extends the well-known fundamental properties of different to the corresponding properties of s-different in general case. Finally he carries through considerations parallel to those on differentials which were given by Chevalley [Introduction to the theory of algebraic functions of one variable, Math. Surveys, no. VI, Amer. Math. Soc., New York, 1951; these Rev. 13, 64] and proves a genus-theorem of Hurwitz using the s-differential-class for a general finite extension K of an algebraic function field k of one variable where the constant field of K and k are the same. *Y. Kawada* (Princeton, N. J.).

***Roquette, Peter. L'arithmétique des fonctions abéliennes.** Colloque sur les fonctions de plusieurs variables, tenu à Bruxelles, 1953, pp. 69–80. Georges Thone, Liège; Masson & Cie, Paris, 1953.

This paper contains a report of the author's work on the more elementary parts of the theory of abstract fields of abelian functions and references to A. Weil's trace function of the ring of complex multiplications.

O. F. G. Schilling (Chicago, Ill.).

Wirsing, Eduard. Ein metrischer Satz über Mengen ganzer Zahlen. Arch. Math. 4, 392–398 (1953).

Ostmann calls a sequence of integers $a_1<a_2<\cdots$ primitive if there do not exist two sequences $b_1<b_2<\cdots$ and

$c_1 < c_2 < \cdots$ each having more than one element so that the sequence $\{b_i + c_j\}$ consists precisely of the a's (the same a_r may occur several times in the form $b_i + c_j$). The sequence is called totally primitive if every sequence which differs from it only in a finite number of terms is primitive. To every sequence $a_1 < a_2 < \cdots$ the author makes correspond the real number $t = \sum_{r=1}^{\infty} 1/2^{a_r}$. The author proves that almost all sequences are totally primitive, i.e., the set of real numbers which correspond to the non-totally primitive real numbers forms a set of measure 0. *P. Erdös.*

Malyšev, A. V. On the representation of large numbers by positive ternary quadratic forms. Doklady Akad. Nauk SSSR (N.S.) **87**, 175–178 (1952). (Russian)

Suppose h is a given odd positive integer. The author sketches a proof of the existence of positive numbers m_0 and c depending only on h and having the following property: If $m \equiv 1, 2, 3, 5, 6 \pmod 8$, $(m, h) = 1$, $(-m|p) = 1$ for all primes p dividing h, and $m > m_0$, if l_0 is an integer such that $l_0^2 \equiv -m \pmod h$, and if H is any (Lipschitz) integral quaternion of norm h, then among the $R(m)$ primitive lattice points (x, y, z) on the sphere $x^2+y^2+z^2 = m$ there are more than $cR(m)$ for which $l_0 + xi + yj + zk = HU$, where U is a (Lipschitz) integral quaternion. This result is a stronger form of a lemma used by Linnik in deriving theorems on the existence of representations of large natural numbers by positive ternary quadratic forms [Izvestiya Akad. Nauk SSSR. Ser. Mat. **4**, 363–402 (1940); these Rev. **2**, 348], and in fact the author states the corresponding stronger form of two of Linnik's theorems. However, no mention is made of the fact that a serious error in the proof of Linnik's lemma was discovered and corrected by G. Pall [Amer. J. Math. **64**, 503–513 (1942); these Rev. **4**, 34]. The reviewer was unable to tell from the sketch given whether or not a similar flaw occurs in the proof of the present author.

P. T. Bateman (Urbana, Ill.).

Linnik, Yu. V., and Malyšev, A. V. On integral points on a sphere. Doklady Akad. Nauk SSSR (N.S.) **89**, 209–211 (1953). (Russian)

Suppose q is a given odd prime number and λ is a given positive number less than unity. Using a result of E. M. Wright [Quart. J. Math., Oxford Ser. **7**, 230–240 (1936), Theorem 1] and the main result of the paper reviewed above, the authors prove that there exist positive numbers c and m_0, depending on q and λ, such that if $m \equiv 1, 2, 3, 5, 6 \pmod 8$, $(-m|q) = 1$, and $m > m_0$, then on any segment of the sphere $x^2+y^2+z^2 = m$ which has surface area greater than $\lambda 4\pi m$ there are more than $cR(m)$ primitive lattice points, where $R(m)$ is the total number of primitive lattice points on the sphere mentioned. This is related to another result of Wright [Proc. London Math. Soc. (2) **42**, 481–500 (1937), Theorem 2]. *P. T. Bateman* (Urbana, Ill.).

Malyšev, A. V. On the representation of numbers by positive ternary quadratic forms. Doklady Akad. Nauk SSSR (N.S.) **89**, 405–406 (1953). (Russian)

The author shows that the following theorem can be deduced rather easily from a theorem on the representations of large positive integers by sums of three squares which is stated in the paper reviewed second above. Let $f(x, y, z)$ be a positive ternary properly primitive quadratic form with invariants $[k, 1]$, where k is odd, suppose f belongs to the genus such that $(f|p) = (-1|p)$ for all primes p dividing k, and let g be an arbitrary odd positive integer. Suppose m, x_0, y_0, z_0 are integers satisfying the conditions: (i) $m \equiv 1, 2, 3, 5, 6 \pmod 8$; (ii) $(m, kg) = 1$; (iii) $m \equiv f(x_0, y_0, z_0) \pmod{8kg}$; (iv) $(m|q) = (-1|q)$ for each prime q dividing g; (v) $m > m_0(k, g)$. Then there are more than $c(k, g)h(-m)$ primitive representations of the number m by the form $f(x, y, z)$ such that $x \equiv x_0$, $y \equiv y_0$, $z \equiv z_0 \pmod g$. Here $c(k, g)$ and $m_0(k, g)$ are positive numbers depending only on k and g, and $h(-m)$ is the number of classes of binary quadratic forms $au^2+buv+cv^2$ (with integral coefficients) such that $b^2 - 4ac = -4m$.

primitive

P. T. Bateman (Urbana, Ill.).

Swinnerton-Dyer, H. P. F. Inhomogeneous lattices. Proc. Cambridge Philos. Soc. **50**, 20–25 (1954).

Let K be an open non-empty set of points in Euclidean n-space and Λ an inhomogeneous lattice $x_i = \sum_{j=1}^{j=n} a_{ij}u_j + x_i^0$, $1 \leq i \leq n$, and $u_1, u_2, \cdots, u_n$ integral. Λ is defined to be K-admissible if it does not contain any point of K. A boundary point P of K is defined to be exterior if it is the limit of points of K on the segment OP. Let R_0 denote the closure of the set of these points and where $\epsilon > 0$ let R_ϵ denote the set of points $(1+\lambda)P$, $P \in R_0$, $0 \leq \lambda \leq \epsilon$. The author shows, where $\Delta(K)$ is the lower bound of the determinants of the K-admissible lattices, that $\Delta(K)$ is the lower bound of those K-admissible lattices which have a point in R_ϵ where ϵ is an arbitrary positive number. K is defined to be automorphic if a number c exists so that, for X within K, a homogeneous linear unimodular substitution Ω exists with $\Omega K = K$ and $|\Omega X| < c$. By use of his first result the author shows that if K is automorphic and has the property that every line through a point of R_0 contains points of K then if $\Delta(K)$ is finite a K-admissible lattice exists with determinant $\Delta(K)$ which contains a point of R_0. It is stated that examples exist which show that the result cannot be improved. *D. Derry* (Paris).

Rogers, C. A. Almost periodic critical lattices. Arch. Math. **4**, 267–274 (1953).

The lattice Λ is said to be periodic for a group G of linear transformations if there exists a compact set H contained in G with the property that, if Ω is any linear transformation of G, there is a Θ in H such that $\Theta\Omega\Lambda = \Lambda$. The lattice Λ is said to be almost periodic for a group G of linear transformations if, for every neighborhood of Λ in the space of lattices, there exists a compact set H contained in G, with the property that, if Ω is any linear transformation of G, there is a Θ in H such that $\Theta\Omega\Lambda$ lies in the given neighborhood of Λ. The author shows by a direct application of results of Gottschalk [Ann. of Math. (2) **47**, 762–766 (1946); these Rev. **8**, 159] that every automorphic star body S with $\Delta(S) < \infty$ possesses a critical lattice which is almost periodic for its group of automorphisms. Further results in case S or G satisfy certain conditions. [For the above terminology cf. H. Davenport and C. A. Rogers, Philos. Trans. Roy. Soc. London. Ser. A. **242**, 311–344 (1950); these Rev. **12**, 394.] *J. F. Koksma* (Amsterdam).

Cugiani, Marco. Sui punti esclusi dalle coperture dell'insieme razionale. Boll. Un. Mat. Ital. (3) **8**, 294–300 (1953).

Let p_N/q_N be the Nth of the irreducible fractions, $0 \leq p/q \leq 1$ arranged according to $p+q$ and subsidiarily according to p/q. For any $\epsilon > 0$ the author defines a set $\mathfrak{s}(\epsilon)$ consisting substantially of the points distant less than $2^{-N}\epsilon$ from some p_N/q_N; the author's definition is slightly different

but much more elaborate. An immediate application of Liouville's theorem shows that to almost all α and in particular for all irrational algebraic α there is an $\epsilon_0=\epsilon_0(\alpha)>0$ such that α non-ϵ $\vartheta(\epsilon)$ for all $\epsilon<\epsilon_0$. The $\vartheta'(\epsilon)$ depending on other orderings of the p/q are also considered. The problem is attributed to W. R. Transue. *J. W. S. Cassels.*

ANALYSIS

Mohr, Ernst. Über die Funktionalgleichung des arithmetisch-geometrischen Mittels. Math. Nachr. 10, 129–133 (1953).

It is shown that the Gauss arithmetic-geometric mean of two numbers is characterized by the functional equation

$$M\left[1+\frac{2t}{1+t^2}, 1-\frac{2t}{1+t^2}\right]=\frac{t^2}{1+t^2}M\left[1+\frac{1}{t^2}, 1-\frac{1}{t^2}\right],$$

normalized to satisfy $M(1, 1)=1$. *E. F. Beckenbach.*

Watson, G. N. Two inequalities. Math. Gaz. 37, 244–246 (1953).

(I) If $n\geqq 2$ and $a, b, \cdots, h$ are n unequal real numbers arranged in descending order of magnitude, and $x>0$, then

$$\begin{vmatrix} x^a & x^b & \cdots & x^h \\ a^{n-2} & b^{n-2} & \cdots & h^{n-2} \\ a^{n-3} & b^{n-3} & \cdots & h^{n-3} \\ \cdot & \cdot & \cdots & \cdot \\ a & b & \cdots & h \\ 1 & 1 & \cdots & 1 \end{vmatrix}$$

is positive for $x>1$ and has the sign of $(-1)^{n-1}$ when $0<x<1$. For $n=3$ this reduces to a fundamental elementary inequality.

(II) Schur's inequality [Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, p. 64] is

$$x^\mu(x-y)(x-z)+y^\mu(y-z)(y-x)+z^\mu(z-x)(z-y)\geqq 0$$

for positive x, y, z, and $\mu\geqq 0$. The author observes that the only known proof is unsymmetrical in the variables and that the theorem is also true for $\mu\leqq -1$ [referring to S. Barnard and J. M. Childs, Higher Algebra, Macmillan, London, 1936, p. 217]. He then proves it (symmetrically) for the remaining interval $-1<\mu<0$, and gives symmetrical proofs for $\mu=1, 2$. *R. P. Boas, Jr.*

Remez, E. Ya. Some questions of Čebyšev approximation in a complex region. Ukrain. Mat. Žurnal 5, 3–49 (1953). (Russian)

This is a complete exposition of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) 77, 965–968 (1951); see also Praci Sičnevoĭ Sesiĭ Akad. Nauk URSR. Dopovidi Viddilu Fiz.-Him. Mat. Nauk 2, 207–214 (1944); these Rev. 13, 99; 7, 520; V. K. Ivanov, Mat. Sbornik N.S. 28(70), 685–706 (1951); 30(72), 543–558 (1952); these Rev. 13, 119; 14, 254]. Let $\mathfrak{G}$ be a bounded set of points $Q=(a_1, a_2, \cdots, a_n, l)$ in $(n+1)$-dimensional complex Euclidean space K^{n+1}. For every $z=(z_1, \cdots, z_n)\in K^n$, let $L(z)=\sup_{Q\in\mathfrak{G}}|l^n-\sum_{k=1}^n a_k z^k|$. The problem is to find points z such that $L(z)$ is a minimum. For many purposes, $\mathfrak{G}$ is taken to be closed and convex and to contain $e^{i\theta}\cdot Q$ along with Q $(0<\theta<2\pi)$. A geometric description of $L(z)$ is first given, and a geometric solution of the problem of minimizing $L(z)$. Certain finite subsets of $\mathfrak{G}$ are introduced, called by the author Čebyšev subsystems. It is shown that the z's minimizing $L(z)$ are a convex set (in the usual real sense) in K^n, which may be a single point. A detailed analysis of the possibilities is given. It is proved that every z_0 minimizing $L(z)$ also minimizes $L(z)$ for every Čebyšev subsystem of $\mathfrak{G}$, with the same value for the minimum. A number of other concepts are introduced and results obtained. *E. Hewitt* (Seattle, Wash.).

Pollard, Harry. Solution of Bernstein's approximation problem. Proc. Amer. Math. Soc. 4, 869–875 (1953).

Let $K(u)$ be continuous on $(-\infty, \infty)$ and $O(|u|^{-n})$ for every positive n. The problem is to decide whether the set $\{u^nK(u)\}$ is fundamental in the space of functions continuous on $(-\infty, \infty)$, vanishing at $\pm\infty$, and normed by the maximum; it was set many years ago by S. Bernstein, and although both necessary conditions and sufficient conditions have been given by various authors, this is the first time that a necessary and sufficient condition has been found which involves no restrictive hypotheses on $K(u)$. The author's conditions are that $K(u)\neq 0$; $\int_{-\infty}^{\infty}(1+u^2)^{-1}\log|K(u)|du=-\infty$; and there exists a sequence of polynomials p_n such that $p_n(u)K(u)\to 1$ and $\|p_nK\|$ is bounded. [At about the same time Bernstein himself established a similar necessary and sufficient condition under additional restrictions on $K(u)$: Doklady Akad. Nauk SSSR (N.S.) 88, 589–592; 90, 124 (1953); these Rev. 14, 977; references to earlier work are given in the paper under review.] *R. P. Boas, Jr.*

Boas, R. P., Jr. Remarks on a moment problem. Studia Math. 13, 59–61 (1953).

Using a result of N. Levinson [Trans. Amer. Math. Soc. 43, 240–257 (1938)] the author generalizes a result of J. G.-Mikusiński [Colloquium Math. 2, 138–141 (1951); these Rev. 13, 214]. Let $(\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers satisfying $\sum 1/|\lambda_n|=\infty$, $\arg\lambda_n\to 0$ and $|\lambda_n-\lambda_m|\geqq|n-m|h$ for some $h>0$; and let $f(t)$ be integrable over the finite interval $0\leqq a\leqq t<b$. Then

$$\int_a^b t^{\lambda_n}f(t)dt=O(|(a+\epsilon)^{\lambda_n}|)$$

for every $\epsilon>0$ implies $f(t)=0$ almost everywhere in (a, b). *A. Dvoretzky* (New York, N. Y.).

Mikusiński, J. G.-, and Ryll-Nardzewski, C. A theorem on bounded moments. Studia Math. 13, 51–55 (1953).

A result of the first author [cf. the paper cited in the preceding review] is generalized as follows: Let β_n be positive numbers satisfying $\sum 1/\beta_n=\infty$ and $\beta_{n+1}-\beta_n>\epsilon>0$ $(n=1, 2, \cdots)$, and let $f(x)$ be integrable over the finite interval $1\leqq x\leqq b$ and satisfy $\int_1^b x^{\beta_n}f(x)dx=O(1)$. Then $f(x)=0$ almost everywhere in (a, b). The method of proof is similar to the one in the cited paper. *A. Dvoretzky.*

Mikusiński, J. G.-. A new proof of Titchmarsh's theorem on convolution. Studia Math. 13, 56–58 (1953).

E. C. Titchmarsh [Proc. London Math. Soc. (2) 25, 283–302 (1926)] proved the following theorem: If f and g are integrable over $[0, T]$ and their convolution vanishes a.e. (almost everywhere) in $[0, T]$ then $f=0$ a.e. in $[0, t_1]$ and $g=0$ a.e. in $[0, t_2]$ with $t_1+t_2\geqq T$. Titchmarsh's proof as well as subsequent proofs rely on the theory of analytic or harmonic functions. The author gives a simple proof based on the theorem quoted in the preceding review. *A. Dvoretzky* (New York, N. Y.).

Mikusiński, J. G.-., et Ryll-Nardzewski, C. Un théorème sur le produit de composition des fonctions de plusieurs variables. Studia Math. **13**, 62–68 (1953).

The theorem of E. C. Titchmarsh quoted in the preceding review was extended to several dimensions by J. L. Lions [C. R. Acad. Sci. Paris **232**, 1530–1532 (1951); these Rev. **13**, 231] using methods of the analytic theory of functions. The authors present a proof of an elementary character using methods similar to those applied in the paper reviewed above to prove Titchmarsh's theorem.

A. Dvoretzky (New York, N. Y.).

Calculus

HAVE

***Duschek, Adalbert. Vorlesungen über höhere Mathematik. Dritter Band. Gewöhnliche und partielle Differentialgleichungen. Variationsrechnung. Funktionen einer komplexen Veränderlichen.** Springer-Verlag, Wien, 1953. ix+512 pp. $9.25.

Velasco de Pando, Manuel. On some algebroid functions related with the systematic integration of differential elements. Las Ciencias. Madrid **17**, no. 1, 5–16 (1952).

The quadrature of every differential of form

$$P(e^{kx})R(\sin x, \cos x)dx,$$

where P is a polynomial and R rational, reduces to elementary functions and to $a(s)=\int_0^s e^{ks} \tan \pi s ds$. When $e^{k/2}$ is rational, an infinite product for $\exp (\pi a)=A(s)$ shows that A satisfies $A^p=M(s)$ in any given finite region. Here P is a suitable integer, $M(s)$ is meromorphic in the region, and both P and M depend on the region. The difference equation $a(1+\alpha)=a(1)+e^k a(\alpha)$ shows that it suffices to tabulate $a(x)$ for $-\frac{1}{2}\leqq x\leqq \frac{1}{2}$. Similar properties are established for the function $b(s)=\pi\int_0^s(s-k)^{-1} \tan (\pi s)\, ds$.

R. M. Redheffer (Los Angeles, Calif.).

Mazzarella, Franco. Condizioni di integrabilità di un differenziale binomio mediante le trascendenti ellittiche. Giorn. Mat. Battaglini (5) **1**(**81**), 199–201 (1953).

Hakala, Reino W. On integration of functions of the form $e^{ax}f(x)$. Math. Mag. **27**, 69–74 (1953).

Furstenberg, Harry. Note on one type of indeterminate form. Amer. Math. Monthly **60**, 700–703 (1953).

Gomes, Marcos Expedito Candido. A real rotation operator. Revista Cientifica **4**, no. 1–2, 39–48 (1953). (Portuguese)

An interpretation of $e^{i\theta}$ in terms of orthogonal unit vectors in the plane.

L. M. Milne-Thomson (Greenwich).

Theory of Sets, Theory of Functions of Real Variables

Denjoy, Arnaud. Les matrices d'ordination de toutes puissances. C. R. Acad. Sci. Paris **236**, 345–348 (1953).

E essendo un insieme ordinato, π una potenza non superiore a quella di E, un elemento a di E è detto "limite di E, anteriormente ad a e per la potenza π", se, qualunque sia l'insieme $e\subset E$, interamente formato d'elementi precedenti a ed avente potenza $\pi'<\pi$, esiste almeno un elemento di E precedente a ma seguente e. Applicando il noto postulato di Zermelo, si dimostra che una siffatta potenza π esiste sempre, qualunque sia a. Un insieme ordinato η, di potenza π, è chiamato "matrice degli ordinamenti della potenza π" quando, comunque si scelgano due insiemi $e\subset\eta$, $e'\subset\eta$, e interamente precedente e', entrambi aventi potenze inferiori a π, esistono in η: sia un elemento precedente e, sia uno seguente e', sia uno compreso fra e ed e'. Si dimostra che, se η, η' sono due matrici di potenze rispettivamente π, π' con $\pi'\leqq\pi$, η' è ordinalmente simile a un sottoinsieme di η. Il complementare d'una matrice rispetto al suo involucro ordinale [cfr. Denjoy, L'énumération transfinie, livre I, Gauthier-Villars, Paris, 1946, p. 23; questi Rev. **8**, 254] contiene un insieme ordinalmente perfetto. Altre interessanti proprietà delle dette matrici sono studiate.

T. Viola.

HAVE

***Denjoy, Arnaud. L'énumération transfinie. Livre II. L'arithmétisation du transfini. Première partie. Les permutations spéciales.** Gauthier-Villars, Paris, 1952. pp. 207–436. 3000 francs.

***Denjoy, Arnaud. L'énumération transfinie. Livre II. L'arithmétisation du transfini. Deuxième partie. Les suites canoniques.** Gauthier-Villars, Paris, 1952. pp. 437–614. 2,500 francs.

Questi due volumi che proseguono l'opera già da tempo felicemente iniziata dall'Autore [v. questi Rev. **8**, 254; **14**, 26], sono ispirati a idee profondamente filosofiche il cui ultimo fine è quello di "convincere che la logica del transfinito della seconda classe non è un'innovazione sensibile su quella dell'Analisi classica: essa guida lo spirito con la stessa sicurezza ed è mezzo a risolvere problemi i cui soggetti, qui come altrove, sono attinti alle sorgenti dell'intuizione" (p. 585). È infatti perfettamente lecito e naturale, secondo l'Autore, negare l'esistenza o la validità dei numeri transfiniti della seconda classe, quando si neghi altresì (come fanno per es. E. Borel ed N. Lusin) l'esistenza dei numeri irrazionali, ad eccezione d'un'infinità numerabile di questi: ma un tale atteggiamento è completamente incoerente in chi, senza riserve, accetti l'esistenza della totalità dei numeri reali e riconosca la validità d'ogni procedimento di passaggio a limite (p. 208).

Per comprendere il problema dell' "Aritmetizzazione del transfinito", occorre richiamarsi al cosidetto "sviluppo neperiano" $x=\sum_n^{1,\infty}(a_n/n!)$ (con $0\leqq a_n\leqq n-1$) d'un qualunque numero x dell'intervallo chiuso $(0, 1)$, sviluppo già introdotto nel vol. I (p. 152) [questi Rev. **8**, 254]. Un tale sviluppo permette di far corrispondere ad ogni x di $(0, 1)$, una ben determinata permutazione P della successione N dei numeri naturali. Ciò è possibile secondo convenzioni diverse, delle quali soltanto due vengono qui utilizzate: indicando con e_n il gruppo (insieme finito) dei primi n numeri naturali, ordinato secondo P, con e_n' ed e_n'' i gruppi ottenuti facendo rispettivamente precedere e seguire e_n dal numero 0, risulta n immediatamente seguente a_n in e_n', precedente a_n in e_n'' [nel cap. IV è utilizzata esclusivamente la prima di queste due convenzioni]. Una siffatta permutazione P è ben ordinata ed è, più precisamente, indicata con $P(\alpha)$ e detta di tipo τ_α (ove α è un numero transfinito di seconda classe), se, qualunque sia il numero ordinale β tale che $1\leqq\beta<\alpha$, v'è uno ed un solo intero positivo $n=n_\beta$ che occupi in P il posto β (cioè esista e sia ben determinato il βesimo intero di P), mentre inversamente ad ogni intero positivo n corrisponda uno ed un solo ordinale $\beta=\beta_n$, tale che $1\leqq\beta<\alpha$ e che n occupi il posto β in P.

Le operazioni inverse permettono di far corrispondere ad ogni permutazione P di N, un ben determinato numero x

di (0, 1). Ma poichè, qualunque sia il transfinito α di seconda classe, esistono infinite permutazioni $P(\alpha)$ di tipo τ_α, si deduce che ad ogni α viene a corrispondere, in virtù delle dette operazioni inverse, tutto un insieme $E(\alpha)$ di numeri x di (0, 1). Un x di (0, 1) appartiene ad $E(\alpha)$, se e solo se ha per corrispondente una permutazione $P(\alpha)$. L'aritmetizzazione del transfinito è lo studio degli insiemi $E(\alpha)$: più precisamente essa è un procedimento atto a far corrispondere ad ogni α una permutazione ben determinata P^α (di tipo τ_α) perfettamente individuabile.—Il problema dell'aritmetizzazione è profondamente studiato nel cap. V, senza che ne venga tuttavia raggiunta la soluzione completa. Il cap. IV ha invece essenzialmente valore propedeutico: in esso è data la definizione, sia costruttiva che diretta, dei numeri transfiniti di seconda classe, per mezzo di somme di particolari permutazioni di N, chiamate "permutazioni speciali".

Il detto problema viene subordinato ad altro, più generale, consistente nel trovare una legge che permetta di far corrispondere ad ogni transfinito α una ben determinata "successione canonica" di ordinali $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$, crescente e tendente ad α. Se α è di seconda specie, una tale successione supera (da un certo indice n in poi) un transfinito $\beta<\alpha$ comunque prefissato [se α è di seconda specie, la successione degenera in un unico termine, cioè in $\alpha-1$]. L'Autore indica la legge cercata, in un opportuno procedimento costruttivo, operante per ricorrenza transfinita, pur dichiarando infine di ignorare se esista, o meno, un primo numero α (di seconda specie) non raggiungibile dal procedimento stesso. Tuttavia il procedimento sembra raggiungere effettivamente tutti i transfiniti suscettibili d'essere utilizzati nelle applicazioni della Analisi matematica. È interessante osservare che, in virtù della legge indicata dall'Autore, la successione canonica $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ tendente ad α è perfettamente determinata dai soli suoi primi due termini α_1, α_2.

Questo trattato si presenta con caratteri di grande originalità e profondità ed è veramente ammirevole per le non comuni difficoltà superate. *T. Viola* (Roma).

Denjoy, Arnaud. Le problème de Souslin. I, II, III. C. R. Acad. Sci. Paris **236**, 435–439, 558–559, 641, (1953).
Denjoy, Arnaud. L'ordination des ensembles. C. R. Acad. Sci. Paris **236**, 981–983 (1953).

Per definizione, un continuo ordinale E è un insieme ordinato ed ordinalmente chiuso [cfr. Denjoy, L'énumération transfinie, livre I, Gauthier-Villars, Paris, 1946, pp. 20–23; questi Rev. **8**, 254], privo di coppie d'elementi consecutivi. Il problema s'enuncia: si può affermare che un continuo ordinale E, tale che ogni famiglia di sezioni disgiunte di E sia numerabile (condizione detta di Souslin), è simile al continuo lineare? L'Autore risolve affermativamente questo problema facendolo rientrare, come caso particolare, in un teor. generale (già enunciato, in via ipotetica, da G. Kurepa ma qui per la prima volta completamente dimostrato) cioè: un insieme ordinato E che verifichi la condizione di Souslin, è simile a un insieme lineare [cioè a un sottoinsieme dell'intervallo (0, 1)].

La dimostrazione si basa sul lemma: nelle ipotesi fatte, se $m_1, m_2, \cdots, m_\alpha, \cdots$ è una successione transfinita d'elementi di E, ordinalmente crescente in E, oppure decrescente in E, gli indici α sono numeri ordinali limitati da un transfinito di seconda classe. In tutto il ragionamento, ha particolare importanza il fatto che la totalità delle successioni ben ordinate d'interi positivi può ridursi ad una famiglia numerabile di successioni numerabili. Una siffatta riduzione, nel senso precisamente richiesto dalla questione trattata, viene ottenuta progressivamente mediante una successione numerabile d'operazioni. L'ultima nota contiene delle considerazioni prevalentemente critiche, relative al problema di Souslin. *T. Viola* (Roma).

Kurepa, Georges. Sur une hypothèse de la théorie des ensembles. C. R. Acad. Sci. Paris **236**, 564–565 (1953).
Kurepa, Georges. Sur un principe de la théorie des espaces abstraits. C. R. Acad. Sci. Paris **236**, 655–657 (1953).

L'Autore completa, in modo molto interessante, alcuni pregevoli risultati da lui ottenuti in precedenti lavori, connessi col problema detto di Souslin (in senso generalizzato) [cfr. le note, qui sopra recensite, di A. Denjoy]. *T. Viola* (Roma).

***Eyraud, Henri. Le théorème de récurrence transfinie.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 41–46. Gauthier-Villars, Paris, 1952.

An unconvincing renewed attempt on the part of the author [see, e.g., Cahiers Rhodaniens no. 3 (1951); these Rev. **14**, 146] to prove a certain proposition which is equivalent to the continuum hypothesis [cf. Neumer, Math. Nachr. **9**, 321–342 (1953), §§4, 5, proposition (R_0); these Rev. **15**, 109]. *F. Bagemihl* (Princeton, N. J.).

Zakon, Elias. On the relation of "similarity" between transfinite numbers. Riveon Lematematika **7**, 44–49 (1954). (Hebrew. English summary)

Two ordinals α, β are similar, $\alpha\sim\beta$, if there exist non-zero ordinals σ, τ such that $\sigma\alpha=\tau\beta$. Typical properties of this relation are: 1) If $\alpha\sim\beta$, then it is possible to determine the factors σ, τ so that one of them is finite; 2) two limit ordinals α, β $(\alpha\geqq\beta)$ are similar if, and only if, $\alpha=\omega^\nu\beta$ for some ordinal ν; 3) if $\alpha\sim\gamma$, $\beta\sim\gamma$, $\alpha\leqq\gamma$, and $\beta\leqq\gamma$, then $\alpha\sim\beta$. An example shows that the inequalities in 3) cannot be eliminated, i.e., the relation $\sim$ is not transitive. *M. Jerison.*

Ginsburg, Seymour. A class of everywhere branching sets. Duke Math. J. **20**, 521–526 (1953).

An ordered set S has "sufficiently many non-cofinal subsets" (we say: S is antidirected, in connexion with directed sets) if for no two-point subset $\{a, b\}$ of S the sets $\{x|x\geqq a, x\in S\}$, $\{y|y\geqq b, y\in S\}$ are cofinal. Any nonvoid $X\subseteq S$ is a maximal residual of S if it is a terminal portion of S and is contained in no greater terminal portion cofinal with X; $F(S)$ denotes the system of all such X's ordered by the dual of the inclusion relation. If S is antidirected, it is totally branching [in the sense that each non-terminal point of S has 2 successors having no common successor; cf. Day, same J. **11**, 201–229 (1944); these Rev. **5**, 231; and Kurepa, Thèse, Paris, 1935 = Publ. Math. Univ. Belgrade **4**, 1–138 (1935), in particular, p. 77; the adjective "non-terminal" is unavoidable]; the converse does not hold. If S is ramified and everywhere (totally) branching, S contains a cofinal antidirected subset (Th. 1). If S is everywhere (totally) branching, $F(S)$ is antidirected (Th. 4). Each antidirected S is cofinally similar to $F(S)$ (Th. 5). If A, B are two cofinally similar everywhere branching sets, $F(A)$, $F(B)$ are isomorphic (Th. 2) (here "everywhere branching" can be erased). The paper terminates with two problems. *G. Kurepa* (Zagreb).

Kurepa, G. Über das Auswahlaxiom. Math. Ann. **126**, 381–384 (1953).

Let $(\bar{K})$ be the principle that each partially ordered set has a maximal subset of incomparable elements. Let (V) be the principle that each set can be simply ordered. The author shows that the axiom of choice is equivalent to the logical product of $(\bar{K})$ and (V). For each positive integer n let $\pi[n]$ be the principle that for each set E containing at least n elements there is a one-to-one mapping f of E onto E such that $\{f(x), f^2(x), \cdots, f^n(x)\}$ is a set of n elements, for each x in E. Let $P[3, \aleph_0]$ be the principle that each infinite set is the union of a family $\{X_a | a \varepsilon A\}$ of disjoint subsets such that each X_a is of power $\leqq \aleph_0$. The author notices that $\pi[3]$ implies $P[3, \aleph_0]$ without the axiom of choice. He then tries to show that $P[3, \aleph_0]$ implies $\pi[3]$ without the axiom of choice. It appears to the reviewer however that his proof uses the axiom of choice when, on page 383, lines 30–34, he selects a particular one-to-one mapping of each set X_a into the positive integers.

S. Ginsburg (Coral Gables, Fla.).

Erdös, P., and Rado, R. A problem on ordered sets. J. London Math. Soc. **28**, 426–438 (1953).

The paper deals with the interconnection between an ordered set and its increasing and decreasing sequences. Let S, tS denote an ordered set and its order-type respectively. For a cardinal a, a^- denotes a or the immediate predecessor of a, according as a is nonisolated or isolated. Property P: If $|S| = \aleph_a$, then $\omega_a \leqq tS$ or $\omega_a^* \leqq tS$ or $\alpha, \alpha^* \leqq tS$ $(\alpha < \omega_a)$. Property P': If $|S| = \aleph_a$, then $\omega_a^* \leqq tS$ or $\alpha \leqq tS$ $(\alpha < \omega_a)$. One has $P \rightleftarrows P'$. Theorem: If for each ordinal n, $2^{\aleph_n} = \aleph_{n+1}$, then a transfinite number a possesses the property P, $a \varepsilon P$, if and only if a is regular. The proof takes 10 pages and is connected with the decompositions of the set $\Omega_2(S)$ of all the 2-point-sets $\subseteq S$ into 2 classes (cf. Lemmas 3, 4). In an addendum it is announced that according to L. Gillman the converse of the preceding theorem also holds. Some useful notations are introduced; e.g., $\{x, y\}_< \subseteq S$ means $x \varepsilon S$, $y \varepsilon S$, $x < y$.

Remarks of the reviewer. The references are not complete. E.g., the reviewer's paper [Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 145–151 (1948); these Rev. **10**, 437] contains implicitly that $\aleph_1 \varepsilon P$ and that not $\omega_1, \omega_1^*, \eta \leqq tS$, if $|S| = \aleph_1$. Further (cf. the hint on p. 439 which is without any reference), the idea of using the superposition of orderings of a same set was, as far as we know, first introduced in 1937 by the reviewer [cf. Revista Ci., Lima **43**, 483–500 (1941), p. 487; these Rev. **3**, 225]; and especially the proof and result in the reviewer's paper, C. R. Soc. Sci. Varsovie **32**, 61–67 (1939) [cf. also C. R. Acad. Sci. Paris **205**, 1196–1198 (1937), §1], are connected with the lemmas 3 and 4, of the paper under review. The negative part of the Theorem (if a^- is singular, then not $a \varepsilon P$) is a corollary of Hausdorff [Math. Ann. **65**, 435–505 (1908), Theorem 14]. The same is true for the converse of the preceding theorem: it is sufficient to consider the sets of all dyadic ω_a-sequences ordered alphabetically.

G. Kurepa (Zagreb).

Watanabe, Hideaki. Une remarque sur l'uniformisation des ensembles analytiques plans. Tôhoku Math. J. (2) **5**, 79–82 (1953).

It is known [see Jankoff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 597–598 (1941); these Rev. **3**, 225] that every plane analytic set can be uniformized by an $A_{\rho\sigma\delta}$. The purpose of this note is to show that $A_{\rho\sigma\delta}$ may be replaced in this theorem by $A_{\tau\delta}$, where A_τ denotes the complement of an A_ρ.

F. Bagemihl (Princeton, N. J.).

Krickeberg, Klaus. Darstellungen oberer und unterer Integrale durch Integrale messbarer Funktionen. Arch. Math. **4**, 432–436 (1953).

The author continues his investigations of upper and lower integrals in very general contexts [cf. Math. Nachr. **9**, 86–128 (1953); these Rev. **14**, 735]. The principal result of this note is that if the upper integral of f exists, then f has a measurable and integrable majorant f^* such that

$$\int_X^* f d\mu = \int_X f^* d\mu$$

for every measurable set X; the function f^* may be characterized in order-theoretic terms.

P. R. Halmos.

Kuipers, L. Continuous and discrete distribution modulo 1. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 340–348 (1953).

Let $f(t)$ be a differentiable function, $0 < t < \infty$. The author proves several theorems about the uniform distribution mod 1 of the sequence $f(n)$. Among others he proves the following theorems: 1) Let $f(t)$ satisfy $|tf'(t)| \leqq M$, $0 \leqq t < \infty$. Then the sequence $f(1), f(2), \cdots$ is uniformly distributed mod 1. 2) Let $f(t)$ be twice differentiable, $f'(t)$ and $f''(t)$ are bounded and $f'(t)$ tends to an irrational number as t tends to infinity. Then the sequence $f(1), f(2), \cdots$ is uniformly distributed mod 1. 3) The sequence $n^{1/2} + \sin n$, $n = 1, 2, \cdots$, is uniformly distributed (mod 1). 4) The sequence $\cos(n + \log n)$, $n = 1, 2, \cdots$, is not uniformly distributed mod 1.

P. Erdös (South Bend, Ind.).

Šanin, N. A. On subsets of a natural sequence of numbers having density. Mat. Sbornik N.S. **31**(73), 367–380 (1952). (Russian)

By the density of a subset A of integers one understands the limit, if it exists, of $\Gamma(A, n)/n$ (as $n \to \infty$), where $\Gamma(A, n)$ denotes the number of elements of A among the first n-integers. Although sets having density (α-sets) do not even form a field the author tries to construct a theory of α sets which would exhibit similarity to the usual theory of Lebesgue measurable sets. In order to state at least one of the results the following terminology is needed: (a) One says that A is "almost contained" in B $(A \subset B)$ if $A - B$ (in the usual point-set theoretic sense) has density 0. (b) A set H is called an "upper limit set" of a collective $\mathfrak{M}$ of α sets if H is an α set and each element of $\mathfrak{M}$ is almost contained in H. One then defines a "minimal upper limit set" in the usual way. The author then proves the following: If $\{D_i\}$ is an ascending sequence of α sets, there exists a set H which is the minimal upper limit set of the collection $\{D_i\}$ and $\alpha(H) = \lim_{i\to\infty} \alpha(D_i)$. Here $\alpha(\ )$ denotes the density of the set in parentheses. A variety of other results is given but it is difficult to reproduce them without further specialized terminology.

M. Kac (Ithaca, N. Y.).

Bonati Savorgnan, Carlo. Sulla derivazione di funzioni composte. Rend. Sem. Mat. Univ. Padova **22**, 258–264 (1953).

A theorem proved by Scorza Toso [same Rend. **21**, 198–201 (1952); these Rev. **14**, 456] for composite functions of the form $f(x(t), y(t))$ is generalized by the author to composite functions of the form $f(x(t), y(t), z(t))$. The statement

of this new theorem is quite analogous to the previous one. Only the assumptions are more restrictive now; namely it is now supposed that $f(x, y, z)$ has first partial derivatives which are continuous with respect to each pair of the variables in the parallelepiped $R(a\leqq x\leqq b, c\leqq y\leqq d, e\leqq z\leqq g)$. The author proves this theorem by means of another analogous theorem (with somewhat less restrictive assumptions) where the parameter t coincides with x. Moreover, he remarks that the further generalizations to functions f of more than three variables are immediate. *A. Rosenthal.*

Džvaršeišvili, A. G. On N. N. Luzin's theorem for functions of two variables. Soobščeniya Akad. Nauk Gruzin. SSR 14, 11–15 (1953). (Russian)

The following analog for two dimensions is given for Lusin's theorem: If $f(x, y)$ is measurable and finite almost everywhere in the square $R_0=[-\pi, \pi; -\pi, \pi]$, there exists a continuous $F(x, y)$ such that $F_\lambda'(x, y)=f(x, y)$ almost everywhere in R_0. Here $F_\lambda'(x, y)=(\lambda)\lim_{h\to 0}(hk)^{-1}\Delta(F, x, y, h, k)$, where (λ) lim is taken under the restriction that $\lambda^{-1}\leqq h^{-1}k\leqq\lambda$, $\lambda\geqq 1$, and

$$\Delta(F, x, y, h, k)=F(x, y)-F(x+h, y)-F(x, y+k)+F(x+h, y+k).$$

No proof is given, but the author indicates how to obtain, using a suitable collection of Cantor step functions, the result: If $\Phi(x, y)$ is continuous on R_0 there exists for every $\epsilon>0$ a continuous F agreeing with Φ on the boundary of R_0 such that almost everywhere $\Delta(F, x, y, h, k)=0$ for sufficiently small h and k and $|\Phi(x, y)-F(x, y)|<\epsilon$ on R_0. It is stated as an application that there exists for any such f a double trigonometrical series which is summable R_λ almost everywhere to $f(x, y)$, the R_λ sum being $(\lambda)\lim_{h\to 0}(hk)^{-2}\Delta^2(R, x, y, h, k)$, where R is the result of integrating the series twice with respect to each variable and $\Delta^2(R, x, y, h, k)=\Delta(\Delta(R, x, y, h, k), x-h, y-k, h, k)$.
G. Klein (South Hadley, Mass.).

Hartman, Philip, and Wintner, Aurel. On continuous area-preserving and Legendre transformations. Amer. J. Math. 76, 87–96 (1954).

Theorems of the following type are proved: Let $u(x, y)$, $v(x, y)$ be continuous in some (x, y)-domain. The transformation T_1: $(x, y)\to(u, v)$ has the properties: T_1 is one-to-one, $u(x, y)$ is an increasing function of y for fixed x, T_1 preserves the orientation. T_1 preserves area, if and only if a function $f(x, u)$ with the following properties exists: f has continuous first partial derivatives, $f_x(x, u)$ is an increasing function of u for fixed x, $f_u(x, u)$ is an increasing function of x for fixed u, $T_1=T_3T_2^{-1}$ where T_2 is the transformation $x=x$, $y=f_x(x, u)$ and T_3 is $u=u$, $v=-f_u(x, u)$.
H. Busemann (Los Angeles, Calif.).

Theory of Functions of Complex Variables

Heffter, Lothar. Einfacher Beweis des Satzes von Looman-Menchoff. Arch. Math. 4, 446–447 (1953).

Let R be the interior of a rectangle with sides parallel to the x- and y-axes, and let $f(z)=u(x, y)+iv(x, y)$ be defined and single-valued in R. The conditions which the author imposes on $f(z)$ in order that $f(z)$ be analytic in R are the following: (1) u and v are absolutely continuous everywhere in R; (2) the partial derivatives u_x, u_y, v_x, v_y exist and are finite everywhere in R; (3) u and v satisfy the Cauchy-Riemann equations everywhere in R. These are not the conditions which Looman and Menchoff intended [cf., e.g., Saks, Théorie de l'intégrale, Warsaw, 1933, pp. 242ff].
A. J. Lohwater (Ann Arbor, Mich.).

Cafiero, Federico. Sulle condizioni sufficienti per l'olomorfia di una funzione. Ricerche Mat. 2, 58–77 (1953).

The author gives sufficient conditions that a complex function $f(z)=u(x, y)+iv(x, y)$ be holomorphic, weakening sufficient conditions previously given by Montel, Looman, Menchoff, Besicovitch, and others [S. Saks, Theory of the integral, 2nd ed., Warsaw-Lwów, 1937]. Thus it is shown that $f(z)$ is holomorphic in Δ provided that u and v are bounded and continuous with respect to x and y separately in Δ, have finite first-order derived numbers except at most on a set of linear measure zero and on a denumerable set of line segments in Δ, and satisfy the condition that their first-order partial derivatives (which exist almost everywhere in Δ) satisfy the Cauchy-Riemann equations almost everywhere in Δ. *E. F. Beckenbach.*

Cafiero, Federico. Un'estensione della formula di Green e sue conseguenze. Ricerche Mat. 2, 91–103 (1953).

The author shows that if $u(x, y)$ and $v(x, y)$ are bounded in a rectangular domain Δ and have finite first-order derived numbers at each point of Δ then a necessary and sufficient condition that the function of rectangles $\int_{FR}udx+vdy$ be absolutely continuous is that the difference $v_x'-u_y'$ (which exists almost everywhere in Δ) be summable there. Further, if this condition is satisfied then

$$\int_{FR}udx+vdy=\iint_R(v_x'-u_y')dxdy$$

for every rectangle R in Δ. As a consequence, the result quoted in the preceding review is established again, but under slightly more restrictive hypotheses.
E. F. Beckenbach (Los Angeles, Calif.).

Mergelyan, S. N. On completeness of systems of analytic functions. Uspehi Matem. Nauk (N.S.) 8, no. 4(56), 3–63 (1953). (Russian)

A clear expository article with full proofs on the closure of $\{1, z, z^2, \cdots\}$ in certain spaces of regular functions. L^2-approximation with and without weight functions in bounded and on unbounded sets is discussed. Most of the results have been announced before. New is a theorem of M. Keldyš which concludes the completeness of polynomials in a domain D from the completeness of rational functions with poles outside $\bar{D}$ under a certain geometrical condition on D. Special attention is given to the closure of polynomials in 'lunes', domains which are homeomorphic to $\{|z|<1, |z-\frac{1}{2}|>\frac{1}{2}\}$. The principal papers summarized are M. M. Džrbašyan, Dissertation, Erevan, 1948; Doklady Akad. Nauk SSSR (N.S.) 62, 581–584 (1948); 66, 1037–1040; 67, 15–18 (1949); 74, 173–176 (1950) [these Rev. 10, 364; 11, 94, 95; 12, 248]; M. Keldyš, Mat. Sbornik N.S. 5(47), 391–401 (1939); 16(58), 1–20 (1945); these Rev. 6, 64; C. R. (Doklady) Acad. Sci. URSS (N.S.) 4, 171–174 (1936); 30, 778–780 (1941) [these Rev. 3, 114]; A. I. Markuševič, Dissertation, Moscow, 1934; C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 262–264 (1944) [these Rev. 6, 179]; A. Šaginiyan, ibid. 27, 318–320 (1940); 45, 50–52 (1944); 48, 11–14 (1945); Izvestiya Akad. Nauk SSSR. Ser. Mat. 5, 285–296 (1941); Dissertation, Moscow, 1945; Doklady Akad. Nauk Armyan. SSR 5, 97–100 (1946) and

several other papers in this journal which were not available for review [these Rev. 7, 64, 285; 8, 455].
W. H. J. Fuchs (Ithaca, N. Y.).

Noble, M. E. The consistency of cardinal series. Proc. Cambridge Philos. Soc. 50, 139–142 (1954).

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences such that $|\alpha_n-n|\leqq D$, $|\beta_n-n-\lambda|\leqq D$, where $D<\pi^{-1}\log 2$, and let

$$G(x)=\lim_{N\to\infty}(x-\alpha_0)\prod_{-N}^{N}{}'\left(1-\frac{x}{\alpha_n}\right),$$

with $H(x)$ defined similarly with zeros β_n. The author proves that if $\sum|A_n|^2|n|^{4D}<\infty$, we can define

$$f(x)=G(x)\sum_{-\infty}^{\infty}\frac{A_n}{(x-\alpha_n)G'(\alpha_n)},$$

and then we also have

$$f(x)=\lim_{N\to\infty}H(x)\sum_{-N}^{N}\frac{f(\beta_n)}{(x-\beta_n)H'(\beta_n)}.$$

This is a generalization of the consistency property of the cardinal series ($\alpha_n=n$, $\beta_n=n+\lambda$) [for the theory and references in this case, see J. M. Whittaker, Interpolatory function theory, Cambridge, 1935, pp. 68ff.]. The proof uses Fourier transforms and biorthogonality properties of $\{e^{i\alpha_n x}\}$.
R. P. Boas, Jr. (Evanston, Ill.).

Tsuji, Masatsugu. On the converse of Abel's theorem. J. Math. Soc. Japan 5, 81–85 (1953).

The author gives a new proof of the following theorem of Hardy and Littlewood [Proc. London Math. Soc. (2) 22, 254–269 (1923)]: If $n|a_n|<K$ and $f(x)=\sum a_nx^n\to 0$ as $x\to 1$ along some path C interior to the unit circle, then $\sum a_n=0$. In dealing with the difficult case where C is not a Stolz path, the new proof achieves considerable economy by means of the following lemma: Let D be a simply connected domain bounded by a segment AB of the positive real axis and by a Jordan arc C which connects A and B and lies in the angular domain $0<\arg z<\alpha$. If $f(z)$ is regular in the closure of D, and $|f(z)|$ is bounded by M on AB and by m ($m<M$) on C, then $|f(z)|\leqq M^{1-\theta/\alpha}m^{\theta/\alpha}$ at all $z=re^{i\theta}$ in D.
G. Piranian (Ann Arbor, Mich.).

Rajagopal, C. T. On inequalities for analytic functions. Amer. Math. Monthly 60, 693–695 (1953).

The author proves the following theorem. Suppose $f(z)=f(0)+\sum_{n=1}^{\infty}a_nz^n$ is regular and $f(z)\neq 0$, $|f(z)|<1$, in $|z|<1$. Then

$$|a_n|\leqq 2/e,\quad k\leqq n\leqq 2k-1,\quad\text{and}\quad |f'(re^{i\theta})|\leqq 2/e(1-r^2).$$

Both results are sharp. The proof occurs by considering $F(z)=\log f(z)$ which has a negative real part in $|z|<1$.
W. K. Hayman (Exeter).

Combes, Jean. Sur la détermination des fonctions analytiques par des conditions imposées à leurs dérivées successives. C. R. Acad. Sci. Paris 237, 1482–1484 (1953).

The author announces a number of results on the problem of the determination of an analytic function of a given class by means of the values of the numbers $f^{(n)}(0)$, $n\neq D_k$, and $f^{(D_k)}(z_k)$. His proofs are said to depend on systems of linear equations in infinitely many unknowns. A typical result is as follows. Let $D_n=d_1+\cdots+d_n$, and $|z_n|\leqq 1$. Then if $(\sum_1^n d_j\log d_j)/D_n\log D_n\to k<1$, $f(z)$ cannot be an entire function, not a polynomial, of order less than $1/(1-k)$. The author has also studied the question of the existence of an analytic function in $|z|<R$, $R>1$, with $f^{(n)}(0)=\alpha_n$ $(n\neq D_k)$, $f^{(D_k)}(1)=\beta_k$. [His methods and results seem to have contact with those of P. Davis, Duke Math. J. 20, 345–357 (1953); these Rev. 15, 207.]
R. P. Boas, Jr. (Evanston, Ill.).

Putnam, C. R. On the non-periodicity of the zeros of the Riemann zeta-function. Amer. J. Math. 76, 97–99 (1954).

It is proved that the sequence of the consecutive positive zeros of $\zeta(\frac{1}{2}+it)$ does not contain any periodic subsequence.
E. C. Titchmarsh (Oxford).

Hiong, King-Lai. Sur les fonctions holomorphes admettant des valeurs exceptionnelles. Ann. Sci. Ecole Norm. Sup. (3) 70, 149–180 (1953).

The author considers functions $f(z)$ regular in $|z|<1$,

$$f(z)=c_0+c_kz^k+\cdots\quad(c_0,\ c_k\neq 0;\ c_0\neq 1),$$

which have at most p zeros and for which 1 is an exceptional value in the sense of Picard-Borel or a defective value in the sense of Nevanlinna of defect $\delta(1)\geqq L>0$. [For such functions the Nevanlinna theory gives

$$\log|f(z)|<C(L)(1-r)^{-1}\log\{2/(1-r)\}$$

for $r_0<r<1$, where, however, $r_0=r_0(f)$; compare the inequality for $T(r)$ on p. 152 of R. Nevanlinna, Le théorème de Picard-Borel et . . . , Gauthier-Villars, Paris, 1929.] The author's objective is to obtain an estimate for $\log|f(z)|$ which depends only on c_0 and r (p and L being kept constant), as one has in the case of Schottky's theorem. He announced such a result in a previous paper [C. R. Acad. Sci. Paris 236, 1628–1630 (1953); these Rev. 14, 859]. In the present paper he proves the result for $r_0<r<1$ (p. 159), with an r_0 which in the opinion of the reviewer still depends on $f(z)$ (see formulas (29) and (30)).
J. Korevaar.

Bagemihl, F., Erdös, P., et Seidel, W. Sur quelques propriétés frontières des fonctions holomorphes définies par certains produits dans le cercle-unité. Ann. Sci. Ecole Norm. Sup. (3) 70, 135–147 (1953).

Let $f(z)$ be analytic in $|z|<1$. Then $f(z)$ cannot approach ∞ uniformly, as $|z|\to 1$. However, various authors have previously constructed functions $f(z)$, analytic in $|z|<1$, which approach ∞, as $|z|\to 1$ in a more or less restricted manner. The authors give an elementary example of this type of function: Let $\{n_j\}$ be an increasing sequence of positive integers, and let $Q_j(z)$ be the polynomial of degree n_j with $Q_j(0)=1$ and with simple zeros at the points $(1-1/n_j)\exp(2\pi i\nu/n_j)$, $\nu=0,1,\cdots,n_j-1$. Then the function $f(z)=\prod Q_j(z)$ is analytic in $|z|<1$ and has the following properties, provided that the n_j are properly chosen. (i) If Δ is the complement with respect to $|z|<1$ of the union of suitably chosen, but non-overlapping neighborhoods of all the zeros of $f(z)$, then $f(z)\to\infty$ uniformly, as $|z|\to 1$ and z is restricted to Δ. (ii) $f(z)\to\infty$ along almost all radii of $|z|<1$. (iii) There exists a sequence of concentric circles in $|z|<1$ on which $f(z)\to\infty$ uniformly. (iv) There exists a continuum of spirals in $|z|<1$, starting at the origin, mutually disjoint otherwise, along each of which $f(z)\to\infty$ and such that any two spirals separate the zeros of $f(z)$ into two infinite sets. (v) If $M(r)$ denotes the maximum of $|f(z)|$ on $|z|=r$, then $M(r)$ can be made to approach ∞ with any given degree of slowness. (vi) The function $|f(z)|+|f'(z)|\to\infty$ uniformly, as $|z|\to 1$.

In a slightly different manner the authors also construct a function $f(z)$ which has the property (iii), mentioned

above, but which does not approach ∞ along any radius of $|z|<1$. *F. Herzog* (E. Lansing, Mich.).

Macintyre, Sheila Scott. An interpolation series for integral functions. Proc. Edinburgh Math. Soc. (2) 9, 1–6 (1953).

The author discusses the Gontcharoff interpolation series with coefficients $F^{(n)}(a_n)$, where $a_n=\omega^n$ and $|\omega|=1$, $\arg\omega\not\equiv 0 \pmod{2\pi}$. She shows that it represents all entire functions of exponential type less than ρ_1, the modulus of the smallest zero of $\sum\omega^{\frac{1}{2}n(n-1)}z^n/n!$; the constant ρ_1 is the best possible. More precisely, $F(z)$ is represented if $F(z)=O\{e^{\rho_1|z|}\phi(\rho_1|z|)\}$ with $\sum k^{\frac{1}{2}}\phi(k)<\infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Also read the note on p. 1140

Redheffer, R. M. On a theorem of Plancherel and Pólya. Pacific J. Math. 3, 823–835 (1953).

The author obtains various results (most of them known) on entire functions of exponential type which are bounded or almost bounded on a line. He derives most of them from Paley and Wiener's representation theorem for functions which belong to L^2 on the real axis. Examples. (i) Let $F(z)$ be entire, of exponential type a, and let $|F(x)|\leqq|P(x)|$ $(-\infty<x<\infty)$, where $P(x)$ is a polynomial of degree n. Then $F(re^{i\theta})=O[r^n\exp(ar|\sin\theta|)]$, $r>1$. (ii) In addition to the above hypotheses let $F(x)$ and $P(x)$ be real. Then the equation $F(z)=P(z)\cos(az+b)$, b real, has at most $n+1$ non-real roots. [It should be mentioned that the proofs which the author gives for his new results on completeness of the set (*) $\{\exp(\pm i\lambda_n x)\}$, λ_n complex, are incomplete. In his proofs the author assumes that whenever the set (*) has finite deficiency in $L^2(-a,a)$, the product $\prod(1-z^2/\lambda_n^2)$ represents an entire function of exponential type a. The reviewer doubts if this is true; it would be worth investigating how much is true.] *J. Korevaar* (Madison, Wis.).

Lehto, Olli. On an extension of the concept of deficiency in the theory of meromorphic functions. Math. Scand. 1, 207–212 (1953).

Nevanlinna's first fundamental theorem states that

$$(1)\qquad m(r,a)+N(r,a)=T(r)+O(1)$$

and when $T(r)$ is unbounded the deficiency $\delta(a)$ of $f(z)$ for the value a is defined as $\delta(a)=1-\limsup_{r\to R}[N(r,a)/T(R)]$. To extend these concepts when $T(R)$ is bounded the author obtains a result (2) analogous to (1). Let $f(z)$ be a non-constant meromorphic function in $|z|<R\leqq\infty$, and let G be an arbitrary domain whose boundary is of positive capacity. Let $g^+(w,a,G)$ equal the Green's function $g(w,a,G)$ of G with pole at $w=a$ or zero, according as w belongs to G or to the complement of G. Define the function $\phi(r,a)$ by

$$\phi(r,a)=\frac{1}{2\pi}\int_0^{2\pi}g^+(f(re^{i\varphi}),a,G)d\varphi.$$

Then, for every a in G, (2) $\phi(r,a)+N(r,a)=P(r,a)$, where $P(r,a)$ is the least harmonic majorant of $N(r,a)$ in G. Because of the analogy between (1) and (2) the author takes as the extension of the notion of deficiency the definition $\delta(a)=1-N(R,a)/P(r,a)$. *M. S. Robertson.*

Zhang, Ming-Yng. Ein Überdeckungssatz für konvexe Gebiete. Acad. Sinica Science Record 5, 17–21 (1952). (Chinese summary)

For the special case of convex maps, the author obtains the precise value of Bloch's constant. Namely, it is shown that if $w=z+a_2z^2+\cdots$ gives a schlicht map of the interior of the unit circle on a convex domain D, then there is somewhere on D a circle of radius $\geqq\pi/4$. The extreme value is attained by a map on a parallel strip, and it is suggested but not proved that this is the only extremizing map.

E. F. Beckenbach (Los Angeles, Calif.).

Slobodeckiĭ, L. N. On a problem of the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) 92, 235–238 (1953). (Russian)

Let B be a region in the ζ-plane of finite connectivity and let Σ be the class of functions univalent and meromorphic in B of the form $F(\zeta)=1/(\zeta-z)+\cdots$ or $F(z)=z+\cdots$ according as the pole z is finite or infinite. Let

$$j_\theta(\zeta,z,\zeta')=1/(\zeta-z)+\cdots,\quad j_\theta(\zeta',z,\zeta')=0,$$

be the function which maps B onto the plane cut by arcs of logarithmic spirals $\Im(e^{-i\theta}\log w)=c$, and put

$$p(\zeta,z,\zeta')=j_0^{1/2}(\zeta,z,\zeta')j_{\pi/2}^{1/2}(\zeta,z,\zeta')$$

and

$$q(\zeta,z,\zeta')=j_0^{1/2}(\zeta,z,\zeta')/j_{\pi/2}^{1/2}(\zeta,z,\zeta').$$

It is proved that if $F(z)\in\Sigma$ and γ_{ij} are arbitrary then

$$(*)\qquad \Re e^{-2i\theta}\sum_{i,j=1}^n\gamma_{ij}\log\frac{F(\zeta_i)-F(\zeta_j)}{p(\zeta_i,z,\zeta_j)}\geqq\Re\sum_{i,j=1}^n\beta_{ij}\log q(\zeta_i,z,\zeta_j)$$

where the β_{ij} are certain functions of the γ_{ij}. When B is the region $|\zeta|<1$, the functions p and q are explicitly known and (*) reduces to a generalized form of the distortion theorem due to Goluzin [Mat. Sbornik N.S. 23(65), 353–360 (1948); these Rev. 10, 602]. *A. W. Goodman.*

Rahmanov, B. N. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) 91, 729–732 (1953). (Russian)

From the set of functions univalent in the unit circle the author selects certain subsets defined by geometrical properties of the image regions. Nine theorems are stated without proof, a typical one being that if $\varphi_k(z)\in P_\alpha$, then $n^{-1}\sum_{k=1}^n\varphi_k(z)\in P_\alpha$. *A. W. Goodman* (Lexington, Ky.).

Fel'dman, Ya. S. Some estimates for p-valent functions. Doklady Akad. Nauk SSSR (N.S.) 92, 239–242 (1953). (Russian)

Let $f(z)=z^p+\sum_{n=p+1}^\infty c_nz^n$ be p-valent in $|z|<1$ and suppose that $m_0(r)\geqq\sum_{n=p}^\infty|c_n|^2r^n$, where $m_0'(r)$ is nondecreasing. It is proved that if $0<\lambda<2$ and $0<r<1$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^\lambda d\theta\leqq\lambda p^{1-\frac{1}{2}\lambda}\int_0^r m_0'(r^2)^{\frac{1}{2}\lambda}r^{\lambda-1}dr.$$

This result is combined with known bounds for $|f(z)|$ [Hayman, Tech. Rep. no. 11, Navy Contract N6-ori-106 Task Order 5. Stanford Univ., Calif., 1950; these Rev. 12, 401] to obtain bounds for $|c_n|$ when $f(z)$ is p-valent and k-wise symmetric. In particular, when $k=1$, $|c_n|\leqq\frac{1}{2}en^{2p-1}$.

A. W. Goodman (Lexington, Ky.).

Komatu, Yûsaku. Identities concerning canonical conformal mappings. Kōdai Math. Sem. Rep. 1953, 77–83 (1953).

The author exhibits various examples of analytic functions which map a given multiply-connected domain onto domains whose boundaries lie on the real axis.

Z. Nehari (St. Louis, Mo.).

Ozawa, Mitsuru. On the conditions of univalency of conformal mapping. Kōdai Math. Sem. Rep. **1953**, 84–86 (1953).

The author studies the formal relations between the necessary and sufficient conditions for univalence due to Grunsky [Math. Z. **45**, 29–61 (1939)] and the equivalent conditions given by Bergman and Schiffer [Compositio Math. **8**, 205–249 (1951); these Rev. **12**, 602].

Z. Nehari (St. Louis, Mo.).

Stallmann, Friedemann. Konforme Abbildung gewisser Kreisbogenvierecke als Eigenwertproblem. Math. Z. **59**, 211–230 (1953).

This paper is concerned with the conformal mapping of a quadrilateral Q, consisting of an angular space $|\arg z|<\delta$ with a semi-circular cutout, onto a half-plane. By classical results, the mapping function is obtained by integrating a Lamé equation which contains an accessory parameter λ depending on the geometric shape of Q. F. Klein had suggested fixing λ by means of certain eigenvalue problems which can be easily interpreted in terms of the geometric shape of Q. The author carries out this program in detail and develops an algorithm for the computation of λ.

Z. Nehari (St. Louis, Mo.)

Jenkins, James A. Another remark on "Some problems in conformal mapping". Proc. Amer. Math. Soc. **4**, 978–981 (1953).

The author gives an example of a triply connected domain D, whose three boundary components do not reduce to points and a triply connected subdomain D', having the same topological situation, such that in every conformal map of D' into D which preserves the topological situation some points of each of the three boundary continua of D' correspond to interior points of D. This answers a question raised verbally by Beurling. *W. K. Hayman* (Exeter).

Belinskiĭ, P. P. On distortion in quasi-conformal mappings. Doklady Akad. Nauk SSSR (N.S.) **91**, 997–998 (1953). (Russian)

Using terminology of Lavrentiev [C. R. Acad. Sci. Paris **200**, 1010–1012 (1935)], the author calls a quasi-conformal mapping q-quasi-conformal if the characteristic function $p(z)$ is bounded by a constant q. It is shown that if $w=f(z)$, $f(0)=0$, is q-quasi-conformal and maps $|z|\leqq 1$ onto $|w|\leqq 1$, then (1) $|f(re^{i\theta})|\leqq\rho(r,q)$, where $\rho=\rho(r,q)$ is uniquely defined by the relation $K'(\rho)/K(\rho)=q^{-1}K'(r)/K(r)$, $K(r)$ and $K'(r)$ denoting, respectively, elliptic integrals of the first kind with moduli $k=r$ and $k'=(1-r^2)^{\frac{1}{2}}$. Furthermore, if an arc of $|z|=1$ of length φ goes over into an arc of $|w|=1$ of length $\tilde{\varphi}$, then (2) $\tilde{\varphi}\leqq\psi(\varphi,q)$, where $\psi=\psi(\varphi,q)$ is determined uniquely from the equation

$$\frac{K'(\sin\frac{1}{4}\psi)}{K(\sin\frac{1}{4}\psi)}=\frac{1}{q}\,\frac{K'(\sin\frac{1}{4}\varphi)}{K(\sin\frac{1}{4}\varphi)}.$$

Conditions under which equality holds in (1) and (2) are discussed. [Reviewer's remark: these theorems are contained in more general results on distortion in pseudo-analytic mappings due to J. Hersch and A. Pfluger, ibid. **234**, 43–45 (1952); these Rev. **13**, 736.] *A. J. Lohwater*.

Belinskiĭ, P. P. Behavior of a quasi-conformal mapping at an isolated point. Doklady Akad. Nauk SSSR (N.S.) **91**, 709–710 (1953). (Russian)

The author announces the following result: Let $w=f(z)$ give a quasi-conformal mapping of

$$0<|z|\leqq 1 \quad\text{onto}\quad 0<|w|\leqq 1,$$

and let the characteristic function (coefficient of dilatation) $p(z)$ satisfy the condition

$$\iint\limits_{0<|z|\leqq 1}\frac{p(z)-1}{|z|^2}\,dxdy=A<\infty.$$

Then $\lim_{z\to 0}f(z)=w_0$ exists, $f(z)$ is monogenic at $z=0$, and $\lim_{z\to 0}(f(z)-w_0)z^{-1}\neq 0,\ \infty$. *A. J. Lohwater*.

***Iwasawa, Kenkichi. Daisu kansu-ron. [Theory of algebraic functions.]** Iwanami Shoten, Tokyo, 1952. i+2+356 pp. 700 yen.

The book begins with a long introduction which is indeed more than an introduction and is an excellent historical survey of the theory of algebraic functions of one variable, from analytical, algebraic geometrical, and algebro-arithmetical view points. Then the first chapter gives a preliminary account of valuations and their prolongations. In the second chapter, fields of algebraic functions (of one variable over an abstract field) are introduced and fundamental notions like prime divisors, (additive) idèles (=repartitions in the sense of Chevalley [Introduction to the theory of algebraic functions of one variable, Math. Surveys, no. VI, Amer. Math. Soc., New York, 1951; these Rev. **13**, 64]=valuation vectors in the sense of Artin [Algebraic numbers and algebraic functions, I, Inst. Math. Mech., New York Univ., 1951; these Rev. **13**, 628]) and genus are defined. Adopted are the definition of differentials and the proof of the Riemann-Roch theorem by Weil [J. Reine Angew. Math. **179**, 129–133 (1938)]. Under the assumption that the constant field k is perfect, Hasse's [ibid. **172**, 55–64 (1934)] definition of differentials is given and its equivalence with Weil's is shown under the further assumption that k is algebraically closed. Assuming, moreover, k to be of characteristic 0, Riemann's formula on the relationship of genus and ramification indices is given. The notions of divisor classes, Weierstrass points, differentials of 1st, 2nd and 3rd kinds, and rational, elliptic and hyperelliptic fields are briefly discussed. The third chapter gives the theory of Riemann surfaces, introducing them following Weyl and Radó. Existence theorems for functions and differentials are proved by Weyl's [Duke Math. J. **7**, 411–444 (1940); these Rev. **2**, 202] method of orthogonal projections. Covering surfaces and simply connected surfaces are considered, leading to the normal forms of Riemann surfaces. In the fourth chapter, fields of algebraic functions (over the complex number field) are characterized as fields of meromorphic functions over compact Riemann surfaces. The topology of compact Riemann surfaces is also discussed. The fifth, final and most important chapter starts with Abel integrals, their periods and the duality between differentials of 1st kind and 1-homology classes. Riemann's bilinear relations are given in the general setting of differentials not necessarily of 1st kind, and the method of proof employed is the orthodox one of retrosections in contrast to Chevalley's book [loc. cit.]. On discussing so-called additive and multiplicative functions, the theorems of Abel and Jacobi are given, which in effect interpret the divisor-class group of degree 0 as a Jacobian variety and introduce the field of Abelian functions. Riemann's theorem of ϑ-functions and the addition theorem for Abelian functions are given without proof. The chapter closes with considerations on elliptic fields, Hurwitz's formula, unramified extensions and Galois groups, and a reference to Weil's [J. Math. Pures Appl. (9) **17**, 47–87 (1938)] hyperabelian functions.

T. Nakayama (Nagoya).

Trohimčuk, Yu. Yu. On the theory of sequences of Riemann surfaces. Ukrain. Mat. Žurnal 4, 49–56 (1952). (Russian)

Trohimčuk, Yu. Yu. On sequences of analytic functions and Riemann surfaces. Ukrain. Mat. Žurnal 4, 431–446 (1952). (Russian)

These papers are related to recent work of Volkovyskiĭ [Mat. Sbornik N.S. 23(65), 361–382 (1948); these Rev. 10, 365] on the notion of kernel of a sequence of Riemann surfaces. In the first paper the author considers only sequences F_n of Riemann surfaces (over a complex w-plane) from which all branch points have been removed and which have a common disc Q. The definition of a kernel is then the same as that given by Carathéodory [Math. Ann. 72, 107–144 (1912)]. It is pointed out that, although a kernel always exists, it need not be unique; hence a construction given by Carathéodory does not determine a unique kernel. A path p in the w-plane, starting in the projection of Q, is called admissible if almost all F_n (i.e., all but a finite number) can be continued along p, starting in Q. If the continuation is independent of n (for almost all n), p is called normal. It is proved that F_n has a unique kernel F if and only if every normal curve p lies in F; an equivalent condition is given in terms of "normal chains". It is shown that uniform convergence at a point, for a sequence of functions $w=f_n(z)$ mapping subsurfaces of a Riemann surface G onto F_n, implies existence of a unique kernel.

In the second paper other characterizations of the kernel are given for the case in which the F_n are all domains on one Riemann surface R: for example, the kernel is the maximal domain containing Q such that every boundary point is a limit point of a sequence q_n of boundary points of the F_n. For the general case a simple kernel is defined and it is shown that, if the F_n are the Riemann surfaces of the inverses of functions $w=f_n(z)$ meromorphic in a domain G in the z-plane, then the simple kernel is the image of the maximal domain G_0 of uniform convergence of the sequence under the limit mapping $w=f(z)$ (provided f is not a constant). A domain G_A of generalized convergence of type (A) is defined by adding to G_0 certain of its boundary points relative to G. A kernel of type (A) is defined and it is shown that, just as in the preceding case, the limit function f maps G_A on the kernel of type (A). *W. Kaplan.*

Röhrl, Helmut. Fabersche Entwicklungen und die Sätze von Weierstrasz und Mittag-Leffler auf Riemannschen Flächen endlichen Geschlechts. Arch. Math. 4, 298–307 (1953).

It has been proved by S. Bochner [Math. Ann. 98, 406–421 (1927)] and by L. Sario [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 50 (1948); these Rev. 10, 365] that every open Riemann surface of finite genus can be embedded in a closed Riemann surface, and may hence be considered to be an open connected set R^* on a closed surface R. Using this, the author obtains a Faber-Tietz expansion for functions $f(z)$ and differentials $dg(z)$ which are regular in R^* [for such expansions on certain regions on closed surfaces see H. Tietz, Arch. Math. 4, 31–38 (1953); these Rev. 14, 859; and H. Roehrl, ibid. 3, 93–102 (1952); these Rev. 14, 154]. He then gets a Mittag-Leffler type expansion for a meromorphic function or differential on R^* whose prescribed poles do not have an accumulation point in R^* and a Weierstrass type infinite product expansion of a regular function on R^* with prescribed zeros.

G. Springer (Evanston, Ill.).

HAVE

***Rosenbloom, Paul C. Semigroups of transformations of a Riemann surface into itself.** Contributions to the theory of Riemann surfaces, pp. 31–39. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953. $4.00.

The author investigates one-parameter semigroups of analytic mappings (1) $z'=f(z,t)$ of the z-plane into itself, i.e., a system of such transformations closed under the ordinary composition rule. The parameter t is supposed to vary in the complete t-plane and $f(z,t)$ is assumed to be an analytic function of its independent variables z and t. Also the function (2) $t=g(t_1,t_2)$ describing the composition of the mappings belonging to the parameters t_1 and t_2 is assumed to be an entire function of t_1 and t_2. The principal result of the paper is that there are the following six classes of functions f and g satisfying the described conditions:

(a) $f=c$ or $f=z$, g arbitrary;
(b) $g(t_1,t_2)=t_2+c$, $f=\Phi(e^{2\pi i(t/c)})$, $\Phi(\omega)$ analytic for $\omega\neq 0$;
(c) $g=b(t_2-c)+c$, $b^n=1$, $f=\Phi((t-c)^n)$;
(d) $g=a(t_1-b)(t_2-b)+b$, $f=(z-b)(a(t-b))^n+b$;
(e) $g=t_1+t_2-a$, $f=z+b(t-a)$;
(f) $g=t_1+t_2-a$, $f=e^{c(t-a)}(z-b)+b$.

In the particular case $f(z,t)=g(z,t)$ the function f may be interpreted as an analytic composition rule of a semigroup whose elements are represented by points in a complex plane. The following possibilities are then obtained:

$$f(z,t)=c,\ z,\ t,\ z+t-c,\quad\text{and}\quad a(z-b)(t-b)+b.$$

In all formulas a, b, and c are arbitrary constants and n a positive integer.

An essential tool in the derivation of all results is Picard's theorem and generalizations of it. *C. Loewner.*

Jenkins, James, and Morse, Marston. Conjugate nets, conformal structure, and interior transformations on open Riemann surfaces. Proc. Nat. Acad. Sci. U. S. A. 39, 1261–1268 (1953).

Let Q be an open, orientable surface. A pair $[F,G]$ of families of sufficiently regular curves on Q is called a conjugate net if for each sufficiently small neighborhood H there is an interior mapping f defined on H such that F and G are the level lines of the real and imaginary parts of f. The authors derive necessary and sufficient topological conditions on a conjugate net that there be an interior mapping f defined on all of Q such that F and G are the real and imaginary level lines of f. Also considered is the problem of expressing F and G as the real and imaginary level lines of an interior map λ of the universal covering surface of Q with the property that λ is transformed into $a\lambda+b$ under a cover transformation of the covering surface.

H. L. Royden (Stanford, Calif.).

Lelong, Pierre. Sur la représentation d'une fonction plurisousharmonique à partir d'un potentiel. C. R. Acad. Sci. Paris 237, 691–693 (1953).

Jede in einem Gebiete Δ plurisubharmonische (bei K. Oka pseudokonvex genannte) Funktion $V(z_1,\cdots,z_n)=V(z)$ ist insbesondere dort eine subharmonische Funktion der reellen Veränderlichen x_k, y_k, wo $z_k=x_k+iy_k$, und lässt deshalb nach F. Riesz in jedem abgeschlossenen Gebiete $D\subset\Delta$ die Zerlegung

$$(1)\qquad V(z)=H_D(z)-\int_D h(a,z)\,d\mu(a)$$

zu. Dabei ist $h(a, z)$ der Kern der Potentiale in $R^{2n}(x_k, y_k)$. Die Funktion $H_D(z)$ ist harmonisch als Funktion der x_k, y_k in D, $d\mu(a)$ ist die $V(z)$ zugeordnete Belegung. Die Teile der Zerlegung (1) brauchen keinesfalls selbst in Δ plurisubharmonisch zu sein. Verf. untersucht nun insbesondere die Darstellungen (1), falls $V(z)$ im ganzen (offenen Raum) C^n plurisubharmonisch ist (man beachte, dass eine plurisubharmonische Funktion auch den Wert $-\infty$ annehmen darf) und D dementsprechend als C^n gewählt wird.

H. Behnke (Münster).

Lelong, Pierre. Sur l'extension aux fonctions entières de n variables d'ordre fini d'un développement canonique de Weierstrass. C. R. Acad. Sci. Paris **237**, 865–867 (1953).

Im Anschluss an die vorstehend referierte Arbeit wird bei ganzen Funktionen $F(z_1, \cdots, z_n)$ endlicher Ordnung eine Integraldarstellung für $|\log F(z)|$ und für die meromorphen Differentiale $d_z \log F(z)$ angegeben, wo nur auf den Nullstellenflächen $F=0$ integriert wird. Die angegebenen Darstellungen konvergieren gleichmässig in jedem beschränkten Gebiete des Raumes C^n. Dazu vergleiche Kneser, Jber. Deutsch. Math. Verein. **48**, Abt. 1, 1–28 (1938), der ähnliche Integraldarstellungen angibt, deren Konvergenz aber nur in Hyperkugeln gesichert ist, die $F=0$ nicht schneiden.

H. Behnke (Münster).

Yûjôbô, Zuiman. On pseudo-regular functions. Comment. Math. Univ. St. Paul. **1**, 67–80 (1953).

Let f be an interior mapping of a domain D into the plane. Let the components of f have partial derivatives almost everywhere, and let f be of bounded eccentricity except at a set E of measure zero. Finally, let the lim sup $|[f(z+\Delta z)-f(z)]/\Delta z|$ be bounded except on a set E' which is a sum of enumerable sets of finite linear measure. The author calls these functions pseudo-regular. Many theorems for many usual definitions of pseudo-analytic functions go over. He obtains for his definition a pair of distortion theorems, an extension of Montel's theorem on uniform families, some key convergence theorems for univalent functions, a Lindelöf-type theorem, and finally the invariance of types of Riemann surfaces.

C. J. Titus.

Yûjôbô, Zuiman. On the quasi-conformal mapping from a simply-connected domain on another one. Comment. Math. Univ. St. Paul. **2**, 1–8 (1953).

In the paper reviewed above the author defined a class of functions called pseudo-regular functions. He is now able to show that if a simple connected domain Δ, of the hyperbolic type, is mapped into another simply connected domain D by a quasi-conformal mapping $f(z)$ then the correspondence between boundary elements of Δ and of D is one to one. This theorem is then applied to bounded pseudo-regular functions, and he obtains a pair of theorems giving the behavior of functions along curves tending to the boundary of a circle domain and curves tending to infinity in a strip domain.

C. J. Titus (Ann Arbor, Mich.).

Fréchet, Maurice. Propriétés des fonctions para-analytiques à n dimensions. C. R. Acad. Sci. Paris **236**, 2191–2193 (1953).

The author announces further results on properties of para-analytic functions on E^n. They are related to a system of linear, homogeneous partial differential equations with constant coefficients.

C. J. Titus (Ann Arbor, Mich.).

Fréchet, Maurice. Formes canoniques des fonctions para-analytiques à deux et à trois dimensions. C. R. Acad. Sci. Paris **236**, 2364–2366 (1953).

The author obtains a simple form for the differential equations which describe the families A and D of para-analytic functions in two and three dimensions.

C. J. Titus (Ann Arbor, Mich.).

Herrmann, A. Ueber einen Laplace'schen Operator dritter Ordnung. Ann. Univ. Saraviensis **1**, 315–318 (1952).

In a Hausdorff function theory [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. **52**, 43–61 (1900)] of the hypercomplex variable $z=\xi_1e_1+\xi_2e_2+\cdots+\xi_ne_n$; where $e_1=1$, $e_2=\lambda$, $e_3=\lambda^2$, $\cdots$, $e_n=\lambda^{n-1}$ and $a_n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0=0$, the n components $u_j(\xi_1, \cdots, \xi_n)$ of an analytic function $f(z)=\sum_{j=1}^n u_j(\xi_1, \cdots, \xi_n)e_n$ satisfy a first order system of partial differential equations (which plays the rôle of the system of Cauchy-Riemann equations of the usual complex variable theory), and each function $u_j(\xi_1, \cdots, \xi_n)$ satisfies a system of second order partial differential equations (which plays the rôle of Laplace's equation in the usual complex variable theory). The author shows that, by taking $n=3$ and choosing a_0, a_1, a_2, a_3 properly, one obtains among the system of Laplace equations of the corresponding Hausdorff function theory the equation

$$\left(\frac{\partial^3}{\partial x^3}+c\frac{\partial^3}{\partial x^2\partial y}+k\frac{\partial^3}{\partial y^3}\right)u(x, y)=0,$$

considered by E. Lammel [Math. Ann. **122**, 109–126 (1950); these Rev. **13**, 353]; and the equation

$$\left(\frac{\partial^3}{\partial \xi_1^3}+\frac{\partial^3}{\partial \xi_2^3}+\frac{\partial^3}{\partial \xi_3^3}-3\frac{\partial^3}{\partial \xi_1\partial \xi_2\partial \xi_3}\right)u(\xi_1, \xi_2, \xi_3)=0,$$

considered by P. Humbert [J. Math. Pures. Appl. (9) **8**, 145–159 (1929)].

J. B. Diaz (College Park, Md.).

Theory of Series

Newton, Tyre A. A note on the Hölder mean. Pacific J. Math. **3**, 807–822 (1953).

Let $\{S_n\}$ denote the sequence of partial sums of the series $\sum a_n$. Let $H_n{}^0=S_n$ and $H_n{}^k=(n+1)^{-1}\sum_{\nu=0}^n H_\nu{}^{k-1}$ for $n\geqq 0$, $k=1, 2, \cdots$. If $\lim_n H_n{}^k=S$ then $\sum a_n$ is said to be summable Hölder of order k to S, denoted $\sum a_n=S$ (H, k). The author first discusses the properties of certain polynomials solutions $G_j{}^m(n+1)$ of the equations

$$H_n{}^k=\sum_{j=0}^m(-1)^jG_j{}^m(n+1)H_{n-j}^{k+m}$$

for integers $k\geqq 0$ and $m\geqq 0$. In terms of these polynomials a number of theorems are proven regarding Hölder summability of which the following is typical. If $\sum a_n=S$ $(H, q+m)$, $m\geqq 0$, then $\sum_{i=0}^{m-1}(\sum_{j=0}^i(-1)^jG_j{}^m(n+1))a_{n-i}=0$ $(H, q+m)$ is a necessary and sufficient condition that $\sum a_n=S$ (H, q). A Tauberian theorem results if $q=0$. If $q=-m$ a result is obtained regarding negative orders of summation. In the above by definition $H_n{}^{-m}=\sum_{i=0}^m(-1)^iG_i{}^m(n+1)S_{n-i}$ $m\geqq 0$.

V. F. Cowling (Lexington, Ky.).

Agnew, Ralph Palmer. Abel and Riesz transforms of series having bounded partial sums. J. Rational Mech. Anal. **3**, 47–72 (1954).

Let $A(t)=\sum_0^\infty u_kt^k$, $R^r(n)=\sum_{k=0}^n(1-k/(n+1))^ru_k$ be the Abel and the Riesz $R(n, r)$ transformations of a series $\sum u_k$

with bounded partial sums, $|s_k|\leqq M$. Let $n=n(t)\to\infty$ for $t\to 1-$. It is trivial that (*) $\limsup_{t\to 1-}|A(t)-R^r(n)|\leqq H_rM$ with $H_r=2$. The author obtains (*) with smaller values of H_r. If $r>0$ is fixed, the smallest H_r for which (*) holds is $H_r=G_r(q_r)$, where

$$G_r(q)=e^{-q}+\int_0^1|qe^{-qx}-r(1-x)^{r-1}|dx,$$

and q_r is the unique value of q which gives $G_r(q)$ its minimum in $0<q<+\infty$. This value of H_r corresponds to $n(t)=q_r/(1-t)$. The author gives also the asymptotic expression $H_r=.4951055\cdots r^{-1}+O(r^{-2})$ for $r\to\infty$ and computes the numerical values of some of the H_r.

G. G. Lorentz (Detroit, Mich.).

Erdös, P. On the uniform but not absolute convergence of power series with gaps. Ann. Soc. Polon. Math. 25 (1952), 162–168 (1953).

The author proves that given any increasing sequence $(n_i)_{i=1}^\infty$ of positive integers satisfying $\lim\inf(n_j-n_i)^{1/(j-i)}=1$ as $j-i\to\infty$, then there exists a power series $\sum_{i=1}^\infty a_iz^{n_i}$ converging uniformly in $|z|\leqq 1$ and for which $\sum|a_i|=\infty$. Actually the author proves the following stronger result: Under the above conditions there exists a sequence of positive numbers a_i with $\sum a_i=\infty$ such that for almost all t the series $\sum r_i(t)a_iz^{n_i}$ converges uniformly in $|z|\leqq 1$ (here $r_i(t)$ denotes the ith Rademacher function). A construction of the sequence a_i is given and the result is established through combinatorial and probabilistic arguments.

A. Dvoretzky (New York, N. Y.).

Freud, Géza. On a theorem of L. Fejér in the theory of series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 505–506 (1953). (Hungarian)

This is the Hungarian version of a previous paper [Acta Math. Acad. Sci. Hungar. 3, 173–176 (1952); these Rev. 14, 737].

P. Erdös (South Bend, Ind.).

Gál, I. S. New proof of two theorems concerning tauberian reduction of integrals. Math. Ann. 122, 390–399; corrections, 123, 339 (1951).

L'auteur fournit de nouvelles démonstrations de résultats généralisant un théorème de Rademacher qui est le suivant: si $f_N(x)\in L^p$ ($p\geqq 1$) sur (a,b) et si

$$\int_a^b[f_{M+1}(x)+\cdots+f_{M+N}(x)]^2dx=O(N)$$

uniformément en M pour $N=1,2,\cdots$, alors

$$f_1+\cdots+f_N=o(N^{1/2}(\log N)^{3/2+\epsilon})$$

presque partout sur (a,b) [Math. Ann. 87, 112–138 (1922)]. Les généralisations ont été obtenues par l'auteur et Koksma [Nederl. Akad. Wetensch., Proc. 53, 638–653 (1950); ces Rev. 12, 86] et consistent à user dans les hypothèses, non de $O(N)$, mais d'une fonction $\phi(N,M)=\phi(N)\psi(M,N)$ où $\phi(N)/N^{1+a}$ ($a\geqq 0$) ne décroit pas. Les deux théorèmes correspondent respectivement aux cas $a=0$ et $a>0$. L'essentiel de la nouvelle démonstration emploie les développements en séries orthogonales [Menchoff, Fund. Math. 4, 82–105 (1923)].

M. Zamansky (Paris).

***Vernotte, Pierre. Régularité et séries divergentes.** Publ. Sci. Tech. Ministère de l'Air, no. 282, Paris, 1953. 53 pp.+xii. 800 francs.

This is an abbreviated exposition of the subject of the author's earlier books with similar titles [same Publ. no. 207 (1947); 238 (1950); ibid., Notes Tech. no. 45 (1952); 46 (1953); these Rev. 11, 97; 12, 604; 14, 800].

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Tsuchikura, Tamotsu. Absolute Cesàro summability of orthogonal series. Tôhoku Math. J. (2) 5, 52–66 (1953).

Part I. F. T. Wang showed that the convergence of $\sum|a_n|^pn^\lambda(\log n)^{\mu+\epsilon}$ implies the $|C,\alpha|$ summability of the series $\sum a_n\cos nx$ almost everywhere, in the following cases: (i) $p=2$, $\lambda=0$, $\mu=1$, $\alpha>\frac12$; (ii) $p=2$, $\lambda=0$, $\mu=2$, $\alpha=\frac12$; (iii) $p=2$, $\lambda=1-2\alpha$, $\mu=1$, $0<\alpha<\frac12$. He also showed that (iii) is false with $\epsilon=0$ [J. London Math. Soc. 16, 174–176 (1941); Duke Math. J. 9, 567–572 (1942); these Rev. 3, 231; 4, 37]. Here the author extends Wang's results to other values of p, λ, μ, α, and shows, by a method of Paley and Zygmund employing Rademacher functions, that (i) and (ii) are false with $\epsilon=0$ [cf. Zygmund, Trigonometrical series, Warsaw-Lwów, 1935, p. 125].

Part II. G. Sunouchi and the author showed that if $\varphi(t)$ is even and periodic, $\varphi\in L^p$, $p>1$, and if φ satisfies the local condition

$$\left(\frac1t\int_0^t|\varphi(u)|^pdu\right)^{1/p}=O(\log(1/t))^{-1-\epsilon}$$

as $t\to+0$, then the Fourier series of $\varphi(t)$ is summable $|C,1|$ at $t=0$ [Sunouchi, J. Math. Soc. Japan 1, 122–129 (1949); these Rev. 11, 657; Tsuchikura, Math. Japonicae 1, 135–139 (1949); these Rev. 11, 657]. Here the author improves the conclusion to summability $|C,\alpha|$ for $\alpha>1/p$.

L. S. Bosanquet (London).

Berman, D. L. Approximation of periodic functions by linear trigonometric polynomial operations. Doklady Akad. Nauk SSSR (N.S.) 91, 1249–1252 (1953). (Russian)

The author continues his previous work on polynomial operations [same Doklady (N.S.) 85, 13–16 (1952); 88, 6–12 (1953); these Rev. 14, 57, 767]. Let $f(x)$ be any function of period 2π, integrable over $(0,2\pi)$ and let $f_t=f_t(x)=f(x+t)$. The space of such functions will be denoted by E and we associate with it a norm $\|f\|$ [for a precise definition of the norm, see the second of the papers cited]. We consider a linear operation $U(f)=U(f,x)$ transforming every $f\in E$ into a trigonometric polynomial of order $\leqq n$. We assume the existence of a fixed trigonometrical polynomial $\Phi(x)$ of order n such that for every trigonometric polynomial $T(x)$ of order n

$$U(T,x)=\int_0^{2\pi}T(x+t)\Phi(t)dt.$$

An example of such an operation U is

$$\sigma_n(f,x)=\int_0^{2\pi}f(x+t)\Phi(t)dt$$

and the main result of the note is as follows: If K is any subset of the unit sphere $\|f\|\leqq 1$ such that $f\in K$ implies

$f_t \in K$ for all t, then

$$\sup_{f \in K} \|f - \sigma_n(f)\| \leqq \sup_{f \in K} \|f - U(f)\|.$$

Extension to functions of several variables.

A. Zygmund (Cambridge, England).

Shapiro, Victor L. A note on the uniqueness of double trigonometric series. Proc. Amer. Math. Soc. 4, 692–695 (1953).

On considère la série trigonométrique double $\sum a_M e^{iMX}$ où a_M est complexe, $M=(m, n)$, $X=(x, y)$, $MX=mx+ny$, $|M|^2=m^2+n^2$. L'auteur dit que la série converge circulairement si $S_R(X)=\sum_{|M|\leq R} a_M e^{iMX}$ a une limite finie $S(X)$ pour $R\to\infty$, qu'elle est sommable $(C, 1)$ circulairement si $\sigma_R(X)=2R^{-2}\int_0^R S_u(X)udu$ a une limite finie pour $R\to\infty$. Il démontre le théorème d'unicité suivant: Soit $a_M=o(|M|^{1-\epsilon})$ $(\epsilon>0)$, E un ensemble plan dénombrable fermé. Si $\sum a_M e^{iMX}$ est sommable $(C, 1)$ circulairement et a pour somme 0 en tout point et si $\frac{1}{4}a_0(x+y)^2-\lim_{R\to\infty}\sum_{1\leq|M|\leq R}(a_M/|M|^2)e^{iMX}$ est continue sur le complémentaire de E, alors la série est identiquement nulle.

M. Zamansky (Paris).

Babakova, O. I. On a generalization of trigonometric conjugate series. Doklady Akad. Nauk SSSR (N.S.) 91, 1241–1244 (1953). (Russian)

If $F(t)$ in L^2 has Fourier coefficients a_k, b_k, the author introduces two generalized conjugate series,

$$A_hF,\ B_hF \sim \pm \sum_{k=1}^{\infty} \frac{1 \pm h^{2k}}{1 \mp h^{2k}} (a_k \sin kt - b_k \cos kt)$$

(upper signs for A_h, lower signs for B_h). They may be expressed as integral transforms of F, of convolution form with principal values, with kernels expressed by means of theta functions. The author gives two generalizations of known results involving the ordinary conjugate function. (1) If $F(t)$ has a bounded rth derivative, the best approximation by trigonometric sums to A_hF and B_hF is of order n^{-r} with explicit best-possible constants. (2) The mapping function between an annulus $h<|w|<1$ and a near-annulus $q<|z|<e^{\psi(\theta)}$, where $|\psi^{(k)}(\theta)|\leqq\epsilon$, $k=0, 1, 2$, is expressible up to order ϵ^2 by means of one of the generalized conjugate functions.

R. P. Boas, Jr. (Evanston, Ill.).

Watson, W. H. The application of Fourier transforms in physical problems. Trans. Roy. Soc. Canada. Sect. III. 47, 37–46 (1953).

In diesem Vortrag gibt der Verf. eine Uebersicht über jene Gebiete der theoretischen Physik, bei denen die Fourier'sche Integraltransformation mit Vorteil angewendet werden kann. Er illustriert diese Zusammenstellung durch Behandlung einiger Beispiele wie die Fortpflanzung von Wellen, Diffusion von Neutronen, Zerfall von Strahlungen etc. und bespricht insbesondere den Zusammenhang der Fourier'schen Integraltransformation mit der Theorie der Verteilungen von L. Schwartz und ihrer Bedeutung für die Lösung von partiellen Integralgleichungen.

W. Saxer.

Wlodarski, L. Sur une formule de Efros. Studia Math. 13, 183–187 (1953).

Let $F(s)$ denote the Laplace transform of a function $f(t)$. Convenient sufficient conditions on $f(t)$ and on functions $q(s)$ and $v(s)$ of the complex variable s are established under which the inverse transform of $v(s)F[q(s)]$ exists and is equal to $\int_0^\infty f(\tau)W(t, \tau)d\tau$ where $W(t, \tau)=L^{-1}\{v(s)\exp[-\tau q(s)]\}$.

R. V. Churchill (Ann Arbor, Mich.).

Fox, Charles. The inversion of convolution transforms by differential operators. Proc. Amer. Math. Soc. 4, 880–887 (1953).

It is proved that if $a_n>0$, $n=1, 2, \cdots$, $\sum_1^\infty a_n^{-2}<\infty$, $E(u)=\prod_1^\infty(1+u^2a_n^{-2})$, and if

$$K(x, v)=\int_0^\infty uJ_\nu(xu)J_\nu(uv)[E(u)]^{-1}du \quad (\nu\geqq-\tfrac{1}{2}),$$

$$f(x)=\int_0^\infty K(x, v)(xv)^{\frac{1}{2}}\phi(v)dv,$$

then

$$\lim_{n\to\infty}\prod_1^\infty\left[1-a_n^{-2}\left(D^2-\frac{\nu^2-\frac{1}{4}}{x^2}\right)\right]f(x)=\tfrac{1}{2}[\phi(x+0)+\phi(x-0)],$$

provided that $\phi \in L(0, \infty)$ and ϕ is of bounded variation in a neighborhood of x. This result is related to and a partial generalization of work by the reviewer and D. V. Widder [Trans. Amer. Math. Soc. 66, 135–201 (1949); these Rev. 11, 350].

I. I. Hirschman, Jr. (St. Louis, Mo.).

Chakrabarty, N. K. Operational calculus with two variables. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 203–217 (1953).

The author lists some obvious identities involving two-dimensional operational images, and gives examples of these. [He seems to think that simultaneous operational calculus was invented in 1941, and does not appear to know many references between then and now. Simultaneous operational calculus goes back at least as far as Heaviside, there is a chapter devoted to it in the recent book by van der Pol and Bremmer [Operational calculus, Cambridge, 1950; these Rev. 12, 407], and an entire book by Voelker and Doetsch [Die zweidimensionale Laplace-Transformation, Birkhäuser, Basel, 1950; these Rev. 12, 699].]

A. Erdélyi (Pasadena, Calif.).

Srivastava, H. M. On some sequences of Laplace transforms. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 218–228 (1953).

The author computes the two-dimensional operational image of functions such as $x^\alpha y^\beta f(x^\gamma y^\delta)$ (for certain values of α, β, γ, δ),

$$(x+y)^{-1}(xy)^{-\frac{1}{2}}f([x+y]/4xy), \quad (xy)^{-1}(x+y)^{-\frac{1}{2}}f(xy/[x+y])$$

by repeated application of the one-dimensional Laplace integral or its inversion, and gives some extensions to 3 and more variables, and a number of examples. [He states that "the idea (of operational calculus) has been generalized to n variables by P. Delerue" [thesis, 1951]. See the reviewer's remark in the preceding review.]

A. Erdélyi.

Polynomials, Polynomial Approximations

Mohr, Ernst. Ein elementarer Beweis für den Fundamentalsatz der Algebra in Reellen. Ann. Mat. Pura Appl. (4) 34, 407–410 (1953).

Die folgende Note enthält im Stile einer Anfängervorlesung über Differential- und Integralrechnung einen elementaren Beweis des Fundamentalsatzes der Algebra im Reellen.

Author's summary.

Ancochea, Germán. Zeros of self-inversive polynomials. Proc. Amer. Math. Soc. 4, 900–902 (1953).

This is a simple and elegant proof of a theorem of A. Cohn [Math. Z. **14**, 110–148 (1922)]. A polynomial $g(z)$ with its zeros symmetric with respect to the unit circle C has the same number of zeros outside C as its derivative $g'(z)$. The proof uses only Rouché's theorem and the continuity of the zeros of a polynomial as functions of its coefficients.

F. F. Bonsall (Newcastle-upon-Tyne).

Vacca, Maria Teresa. Determinazione asintotica per $n\to\infty$ degli estremi relativi dell' n^{esimo} polinomio di Jacobi. Boll. Un. Mat. Ital. (3) **8**, 277–280 (1953).

Tricomi [same Boll. (3) **8**, 107–109 (1953); these Rev. **15**, 29] has shown that the rth relative maximum of $|P_n(x)|$ approaches the rth relative maximum of $|J_0(x)|$, as $n\to\infty$. The author proves the corresponding statement for $n^{-\alpha}|P_n^{(\alpha,\beta)}(x)|$ (the maxima being counted from $x=1$).

A. Erdélyi (Pasadena, Calif.).

Colucci, Antonio. Generale maggiorazione dei polinomi e delle derivate e una sua conseguenza. Boll. Un. Mat. Ital. (3) **8**, 258–260 (1953).

If a polynomial of nth degree has all its zeros in the circle $|z|\leq\rho$ and its highest coefficient (coefficient of z^n) is a_0, the modulus of the kth derivative is majorized by the expression $k!\binom{n}{k}|a_0|(|z|+\rho)^{n-k}$. Various simple applications are noted. *G. Szegö* (Stanford, Calif.).

Ossicini, Alessandro. Funzioni di Legendre di seconda specie e polinomi ultrasferici. Boll. Un. Mat. Ital. (3) **8**, 304–309 (1953).

The series

$$\sum_{n=0}^{\infty}\frac{(n+\lambda)n!}{\Gamma(n+2\lambda)}Q_{n+\lambda-\frac{1}{2}}(z)P_n^{(\lambda)}(x)P_n^{(\lambda)}(y)$$

where Q is the Legendre function of the second kind and $P_n^{(\lambda)}$ the ultraspherical polynomials, is expressed in terms of elementary functions. Numerous special cases are considered. Another expansion is considered involving Q and the general Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. *G. Szegö.*

Ganzburg, I. M. On certain methods of approximation of summable functions by means of polynomials. Ukrain. Mat. Žurnal **5**, 304–311 (1953). (Russian)

It is well known that if $f\in L_r(0, 2\pi)$ for $r>1$, then for every $k>0$ and almost every x

$$n^{-1}\sum_{\nu=1}^{n}|S_\nu(x;f)-f(x)|^k\to 0,$$

where the S_ν are the partial sums of the Fourier series of f, and that this result fails for $r=1$. It may be verified for functions satisfying $\int_{-\pi}^{\pi}|f(x-h)-2f(x)+f(x+h)|dx=O(|h|^\alpha)$, $0<\alpha\leq 1$. To establish this the author shows that if

$$t_n(x;f)=\tfrac{1}{3}[S_n(x;f)+S_n(x+a_n;f)+S_n(x-a_n;f)],$$
$$a_n=4\pi/3(2n+1),$$

then t_n converges to f at every point of its Lebesgue set. It is remarked that the corresponding summation method for numerical series given by the triangular matrix with entries $a_{nm}=\frac{1}{2}(1+\cos 4\pi m/3(2n+1))$ is stronger than $(C, 1)$.

G. Klein (South Hadley, Mass.).

Ganzburg, I. M. On approximation of functions with a given modulus of continuity by sums of P. L. Čebyšev. Doklady Akad. Nauk SSSR (N.S.) **91**, 1253–1256 (1953). (Russian)

Let H_ω be the class of $f(x)$ defined on $(-1, 1)$ with modulus of continuity not exceeding a function $\omega(t)$ which is convex and satisfies (*) $\sum_{\nu=1}^{n}\nu^{-1}\omega(\nu/n^2)=O\{\omega(1/n)\}$. Let s_n be the partial sums of the expansion of $f(x)$ in terms of the polynomials $\cos\{k\cos^{-1}x\}$, and let $E_n(x)$ be the supremum of $|f(x)-s_n(x)|$ over H_ω. The author establishes the asymptotic formula

$$E_n(x)=2\pi^{-2}\log n\int_0^{\pi/2}\omega\{4u(2n+1)^{-1}(1-x^2)^{\frac{1}{2}}\}\sin u\,du +O\{\omega(1/n)\},$$

where the constant implied by the O depends only on that in (*). This includes, in a sharper form, results for Lip α due to Nikolsky [Izvestiya Akad. Nauk SSSR. Ser. Mat. **10**, 295–322 (1946); these Rev. **8**, 153] and Timan [same Doklady (N.S.) **77**, 969–972 (1951); these Rev. **12**, 823].

R. P. Boas, Jr. (Evanston, Ill.).

Evgrafov, M. A. On completeness of certain systems of polynomials. Mat. Sbornik N.S. **33**(75), 433–440 (1953). (Russian)

Let

$$p(z)=1+a_1z+\cdots+a_kz^k=\sum_{j=1}^{k}(1-z\lambda_j^{-1}),$$
$$|\lambda_1|<|\lambda_2|<\cdots<|\lambda_k|,\quad a_j\neq 0.$$

The author investigates the closure of $\{z^np_n(z)\}$ in the set of functions regular in $|z|<r$. Here $p_n(z)$ is a polynomial of degree k with non-vanishing coefficients, and $p_n(z)\to p(z)$ as $n\to\infty$. It is shown that associated with every λ_j there is a power series $F_j(z)=\sum c_j(n)z^{-n-1}$, convergent in $|z|>|\lambda_j|$, such that the linear functional $l_j(f(z))=\int F_j(z)f(z)dz$ (integration along $|z|=|\lambda_j|+\epsilon$) vanishes for $f(z)=z^np_n(z)$. Theorem. Let $f(z)$ be regular in $|z|<r$, $l_1(f)=\cdots=l_{m-1}(f)=0$, $l_m(f)\neq 0$, and $|\lambda_m|<r$; then there is a convergent expansion $f(z)=\sum_{n=0}^{\infty}b_nz^np_n(z)$ $(|z|<|\lambda_m|)$. If $l_j(f)=0$ for all j with $|\lambda_j|<r$, then the expansion converges in $|z|<r$. *W. H. J. Fuchs.*

Freud, Géza. On strong $(C, 1)$-summability of orthogonal polynomials. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 507–511 (1953). (Hungarian)

This is the Hungarian version of a previous paper [Acta Math. Acad. Sci. Hungar. **3**, 83–88 (1952); these Rev. **14**, 467]. *P. Erdös* (South Bend, Ind.).

Special Functions

HAVE

***Erdélyi, Arthur, Magnus, Wilhelm, Oberhettinger, Fritz, and Tricomi, Francesco G. Higher transcendental functions. Vols. I, II.** Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xxvi+302, xvii+396 pp. \$6.50, \$7.50.

The late Harry Bateman, during his last years, planned an extensive compilation of the "Special Functions". He intended to investigate and tabulate their properties, the inter-relations between them, their representations in vari-

ous forms, their macro- and micro-scopic behavior, and to construct tables of the definite integrals involving them. The whole project was to have been on a gigantic scale; it would have been an authoritative and definitive account of its vast subject.

While much of the material is available, it is not readily accessible, being scattered in books and journals on many fields. The "Guide to the Functions" which Bateman planned would have been invaluable. The project was never completed, and, after his death, the California Institute of Technology and the U. S. Office of Naval Research pooled their resources to continue Bateman's task.

It turned out that no single section of Bateman's work was in a state suitable for immediate publication, and the field was so wide that it appeared essential to narrow it down if anything useful was to be accomplished. It was decided to concentrate on a three-volume work on the Higher transcendental functions (of which the first two are now under review), to be followed by two volumes of tables of integrals. The whole work has been carried out by the staff of the Bateman Manuscript Project, under the directorship of Arthur Erdélyi.

The volumes on the Higher transcendental functions might be described as an up-to-date version of Part II of Whittaker and Watson's "Modern analysis" [4th ed., Cambridge, 1927]. The scope is best indicated by the titles of the chapters: I) The gamma function; II) The hypergeometric function; III) Legendre functions; IV) The generalized hypergeometric series; V) Further generalizations of the hypergeometric function; VI) The confluent hypergeometric function; VII) Bessel functions; VIII) Functions of the parabolic cylinder and of the paraboloid of revolution; IX) The incomplete gamma functions and related functions; X) Orthogonal polynomials; XI) Spherical and hyperspherical harmonic polynomials; XII) Orthogonal polynomials in several variables; XIII) Elliptic functions and integrals.

To attempt to review such a work in detail is quite impossible. Suffice it to say that it is a valuable piece of work well done.

As Erdélyi says in his Introduction, the chapters are of widely varying characters. Some, such as the first chapter, read almost like an advanced textbook, except that proofs sometimes have to be sketched. Other chapters, such as the fifth, which deal with functions which do not appear very frequently in applied mathematics, give a briefer account in a narrative rather than a deductive style. The chapter on Bessel functions tends to concentrate upon those results which are not readily accessible because they appeared after the publication of Watson's "Treatise on the theory of Bessel functions" [Cambridge, 1922]. Each chapter presented its special problems, which had to be treated on its merits. But these differences of treatment in no way detract from the value of the work.

No attempt is made to give a comprehensive bibliography, though at the end of each chapter there is a list of references designed to assist the reader to find further information, and at the end of each volume there is an index of notations and a subject index. *E. T. Copson* (St. Andrews).

Mikusiński, J. G. -. On generalized exponential functions. Studia Math. 13, 48–50 (1953).

Let $\beta_1, \beta_2, \cdots$ be positive numbers satisfying $\sum 1/\beta_n = \infty$ and $\beta_{n+1}-\beta_n>\epsilon>0$ $(n=1, 2, \cdots)$, and put

$$\alpha_n=\epsilon^{-1}\prod\beta_\nu(|\beta_\nu-\beta_n|)^{-1}\exp(-\beta_n/\beta_\nu)$$

with the product extending over all $\nu\neq n$. It is proved that $\alpha_n^{1/\beta_n}\to 0$ and hence, using previous results [Studia Math. 12, 181–190 (1951); these Rev. 14, 39] it is deduced that $1+\sum(-1)^n\alpha_n x^{\beta_n}$ converges for $0\leq x<\infty$ and decreases monotonically from 1 to 0 over this range. *A. Dvoretzky* (New York, N. Y.).

Lehrer, Yehiel. On matrices the elements of which are generalized hyperbolic functions. Riveon Lematematika 7, 71–73 (1954). (Hebrew. English summary)

The generalized hyperbolic functions of order n are the real functions

$$h_j(x)=\sum_{k=0}^{\infty}x^{j+kn}/(j+kn)! \quad (j=0, 1, \cdots, n-1).$$

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$. The author defines the matrices

$$T(x)=[h_{j-i}(x)], \quad T(\alpha; x)=\left[\frac{\alpha_i}{\alpha_j}h_{j-i}(x)\right]$$

(the brackets showing the elements in the ith row and jth column). He shows that $\det T(x)=1$, $T(x)T(y)=T(x+y)$, $T(0)=I$ so that $T(x)$ is isomorphic to an abelian subgroup of the n-dimensional real unimodular group. He also shows that

$$T^{(r)}(\alpha; x)T^{(s)}(\alpha; y)=T^{(r+s)}(\alpha; x+y), \quad [T^{(r)}(\alpha; x)]^n=T(\alpha; x),$$

where the bracketed superscripts denote differentiation with respect to the variable, and concludes that each $T(\alpha; X)$ (α fixed) is isomorphic to $T(x)$, and also that $T^{(1)}(\alpha; 0)$ is the infinitesimal generator of $T(\alpha; x)$. He concludes by pointing out that if A is any $n\times n$ constant matrix for which $A^n=I$ then the elements of e^{Ax} are generalized hyperbolic functions. *A. Erdélyi* (Pasadena, Calif.).

Lehrer, Yehiel. Properties of a system of solutions of a linear differential equation and of the Wronski matrices of this system. Riveon Lematematika 7, 74–76 (1954). (Hebrew. English summary)

Let $f_j(x)$ be those solutions of an ordinary linear differential equation with constant coefficients which satisfy the initial conditions $f_j^{(i)}(0)=\delta_j^i$. The author shows that the Wronskian matrix of the $f_j(x)$ has properties analogous to those of $T(x)$ in the preceding review and he also constructs the matrix analogous to $T(\alpha; x)$ above. *A. Erdélyi.*

HAVE

★Bowman, F. Introduction to elliptic functions with applications. English Universities Press, Ltd., London, 1953. 115 pp. 12 s. 6 d.

"The purpose of this book is to give a short practical introduction to some applications of elliptic functions; it is confined to Jacobian functions; the Weierstrassian function is not even mentioned". (From the author's preface.) The book is written for physicists and engineers, practical computations are in the foreground of interest, and guidance is provided for the effective use of available numerical tables. In the first few chapters only an elementary knowledge of differentiation and integration are required, in the later chapters also elementary complex variable theory. The presentation is clear and explicit. Contents: I. Jacobian elliptic functions. II. Elliptic integrals. III. Applications. Argument real. IV. Argument complex. V., VI., VIII. Conformal representation. VII. Applications. IX. Reduction (of elliptic integrals) to standard form. X. A degenerate hyperelliptic integral. A brief bibliography, tables of formulas, and a subject index are appended. *A. Erdélyi.*

González, Mario O. Memoir on the theory of elliptic functions. Revista Soc. Cubana Ci. Fis. Mat. 3, 39–44 (1953). (Spanish)

This is a historical outline of the origin and development of the theory of elliptic functions and is to be followed by two further introductory chapters on general properties of elliptic functions. The author quotes the contributions of Bernoulli, Fagnano, Euler, Legendre, Abel, Jacobi, Liouville, Hermite, Briot and Bouquet, and Weierstrass. He comments on the analogy of Jacobi's function sn z and Weierstrass' function $\sigma(z)$ with the function sin z and the still stronger analogy of Weierstrass' $\wp(z)=-d^2\log\sigma(z)/dz^2$ with $\operatorname{cosec}^2 z=-d^2\log\sin z/dz^2$, both being meromorphic, even, periodic functions with double poles. Much of the simplicity in Weierstrass' treatment of the elliptic functions is attributed to this analogy, while the more complicated features of this theory are compared to what trigonometry would be like, if $\operatorname{cosec}^2 z$ would be taken as fundamental function. A close analogy has been found to exist between tg z and the elliptic function Tan z introduced by the author [Bull. Amer. Math. Soc. **54**, 833–834 (1948); Actas Acad. Ci. Lima **15**, 93–97 (1952); these Rev. **14**, 271; also Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R. I., 1952, p. 393]; actually, tg z and th z are found to be particular cases of the elliptic function Tan z. In the following chapters of the memoir, the theory of the elliptic functions will be developed, by taking Tan z as fundamental function. *E. Grosswald.*

Agarwal, R. P. Associated basic hypergeometric series. Proc. Glasgow Math. Assoc. 1, 182–184 (1953).

The author proves several identities involving basic hypergeometric series ${}_2\Phi_1$ and the operator D defined by $Df(x)=x^{-1}[f(x)-f(qx)]$, these identities being analogous to differentiation formulas of ordinary hypergeometric series. He also proves that any four associated ${}_2\Phi_1$ satisfy a linear relation with polynomial coefficients. *A. Erdélyi.*

Agarwal, R. P. Some basic hypergeometric identities. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 186–202 (1953).

F. H. Jackson [Quart. J. Math., Oxford Ser. **13**, 69–82 (1942); **15**, 49–61 (1944); these Rev. **4**, 141; **6**, 152] has obtained the basic analogues of some expansions of ordinary hypergeometric series proved by J. L. Burchnall and T. W. Chaundy [ibid. **11**, 249–270 (1940); **12**, 112–128 (1941); these Rev. **2**, 287; **3**, 118]. Since then Chaundy [ibid. **13**, 159–171 (1942); these Rev. **4**, 197] has published more expansions, and the author now provides the basic analogues of these. *A. Erdélyi* (Pasadena, Calif.).

Agarwal, R. P. General transformations of bilateral cognate trigonometrical series of ordinary hypergeometric type. Canadian J. Math. 5, 544–553 (1953).

"The object of this paper is to study general transformations of bilateral cognate trigonometric series in analogy with the ordinary bilateral series introduced by W. N. Bailey [Quart. J. Math., Oxford Ser. **7**, 105–115 (1936)]."

Bilateral trigonometric series here mean series of the form

$$\sum_{n=-\infty}^{\infty}\frac{(\pm1)^n(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_p)_n}\,{\cos\atop\sin}\,(\lambda+2n)\theta,$$

where $(\alpha)_n=\Gamma(\alpha+n)/\Gamma(\alpha)$. With the help of Sears' transformations of unilateral trigonometric series [Proc. London Math. Soc. (2) **53**, 138–157 (1951); these Rev. **13**, 33] the author deduces a number of transformations connecting general and well-poised series. Direct proofs of these transformations are also given, and are based on the use of contour integrals involving gamma functions. *A. Erdélyi* (Pasadena, Calif.).

Nørlund, Niels Erik. Sur les fonctions hypergéométriques. C. R. Acad. Sci. Paris 237, 1466–1468 (1953).

The hypergeometric differential equation

$$(\theta-\gamma_1)\cdots(\theta-\gamma_n)y-z(\theta+\alpha_1)\cdots(\theta+\alpha_n)y=0\quad(\theta=zd/dz)$$

has a single solution with a multiplicative branch-point at $z=1$. The author represents this solution by an integral of the Mellin-Barnes type and indicates how the integral representation may be used to express this solution in terms of the fundamental system belonging to $z=0$ (or $z=\infty$). *A. Erdélyi* (Pasadena, Calif.).

Nørlund, Niels Erik. Sur les fonctions hypergéométriques. C. R. Acad. Sci. Paris 237, 1371–1373 (1953).

It is well known that the hypergeometric differential equation

$$(\theta-\gamma_1)\cdots(\theta-\gamma_n)y-z(\theta+\alpha_1)\cdots(\theta+\alpha_n)y=0$$
$$(\theta=zd/dz,\text{ all }\gamma_i-\gamma_j\text{ non-integer})$$

has $n-1$ linearly independent solutions which are analytic at $z=1$ (and one solution which has a singularity there). The author first defines a suitable fundamental system, $y_1,\cdots,y_n$, of solutions for $|z|<1$, shows that y_i-y_j are solutions analytic at $r=1$ (there are clearly $n-1$ linearly independent among these), and gives an elegant representation of these solutions in series of hypergeometric polynomials. He also indicates briefly the connection between the canonical fundamental system at ∞, and the solutions analytic at $z=1$. *A. Erdélyi* (Pasadena, Calif.).

Hull, T. E., Swanson, C. A., and Trumpler, D. A. Bessel expansions of the confluent hypergeometric functions. Trans. Roy. Soc. Canada. Sect. III. 47, 7–16 (1953).

The multiplication theorems for the Bessel functions are derived by transforming Bessel's equation into a suitable integral equation to which the method of successive approximations is applied. A like procedure leads to an expansion for the Whittaker function $M_{k,m}(x)$ in a series of Bessel functions for k arbitrary complex, x finite complex, and $\Re(m)\geq0$. This duplicates work of Tricomi [Ann. Mat. Pura Appl. (4) **28**, 263–289 (1949); Comment. Math. Helv. **25**, 196–204 (1951); these Rev. **12**, 96; **13**, 343]. A similar series for $M_{k,-m}(x)$, which is an asymptotic expansion for large $|k|$, is also obtained. Related to, but distinct from, these results are asymptotic series in k for the Whittaker functions given by A. Ziebur [Thesis, Univ. of Wisconsin, 1950], these being valid for all x and bounded m. *N. D. Kazarinoff.*

Skovgaard, H. Note on the number of real zeros of the confluent hypergeometric function $F(a;c;x)$. Math. Z. 58, 448–452 (1953).

Tricomi [Math. Z. **52**, 669–675 (1950); these Rev. **12**, 256] gave an elegant representation, by means of a diagram, of Kienast's results [Denkschr. Schweiz. Naturf. Gesell. **57**, 247–325 (1921)] on the number of positive zeros of the confluent hypergeometric series ${}_1F_1(a;c;x)$ for real a, c. The present paper contains a proof of Tricomi's diagram by an extension of Descartes' rule of signs. *A. Erdélyi.*

Chang, Chieh-Chien, Chu, Boa-Teh, and O'Brien, Vivian. Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z. I, II. J. Franklin Inst. 255, 215–236, 319–331 (1953).

The authors' investigations of the asymptotic behavior of $W_{k,m}(z)$ when $k=-k_0n+k_1$, $m=im_0n+m_1+O(n^{-1})$, $z=nz_0$, where k_0, m_0, z_0 are positive real and $n\to\infty$, were described in an earlier paper [J. Rational Mech. Anal. 2, 125–135 (1953); these Rev. 14, 469]. Here a more detailed presentation is given. The authors also make a comparison between Debye's and Langer's methods for obtaining asymptotic formulas. *A. Erdélyi* (Pasadena, Calif.).

Gupta, H. C. Original of $p^{-\lambda}{}_{q+1}F_q(-p^n)$ and its integral representation. Proc. Nat. Inst. Sci. India 19, 691–695 (1953).

The author obtains the inverse operational image of $p^{-\lambda}{}_{q+1}F_q(a_r;b_s;-p^n)$. *A. Erdélyi* (Pasadena, Calif.).

— ***Humbert, Pierre. Calcul symbolique et fonctions hypergéométriques.** Comptes Rendus du Congrès des Sociétés Savantes de Paris et des Départements tenu à Grenoble en 1952, Section des Sciences, pp. 47–51. Gauthier-Villars, Paris, 1952.

The author considers operational relations in which p, the "operational" variable, occurs in one of the parameters of a hypergeometric series, and gives several applications. *A. Erdélyi* (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. V. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 349–357 (1953).

[For parts I to IV see these Rev. 14, 469, 642, 748, 979.] The present installment contains an investigation of the particular case $m=1$ of the "first expansion theorem" (see part III), and, arising out of this investigation, the formulation of the "third expansion theorem" which is an expansion of

$$\lambda^{\alpha}{}_{p+k}F_{q+l-1}\left[\begin{matrix}\gamma_1,\cdots,\gamma_k,\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_{q-1},\delta_1,\cdots,\delta_l;\lambda\zeta\end{matrix}\right]$$

in an infinite series of products of the form $\zeta^r{}_{k+1}F_l(\lambda)_{p+1}F_q(\zeta)$. This expansion is proved under several sets of conditions of validity. *A. Erdélyi* (Pasadena, Calif.).

on order 7/2/54

***Bickley, W. G. Bessel functions and formulae.** Cambridge, at the University Press, 1953. pp. xxx–xl. $.75.

Reprint, without change of page numbers, of the "Summary of notations" and the section on "Functions and formulae" from British Association Mathematical Tables, vol. X, Bessel functions, Part II [Cambridge, 1952; these Rev. 14, 410].

Bochner, S. Bessel functions and modular relations of higher type and hyperbolic differential equations. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 12–20 (1952).

In the Euclidean E_k, $x=(x_1,\cdots,x_k)$, a domain P is taken with the following properties: (a) it does not contain the origin; if $x\in P$, then $tx\in P$ for $0<t<\infty$; (b) P is convex, and, consequently, (c) if $x^1\in P$, $x^2\in P$, then $x^1+x^2\in P$. If the dual E_k of "exponents" $\lambda=(\lambda^1,\cdots,\lambda^k)$ is taken, the points λ for which the inner product $(\lambda,x)=\sum_{j=1}^k\lambda^jx_j\geq 0$ for all $x\in P$ constitute a closed set $\bar{G}$ whose interior G, if not empty, has the same properties as P. (It is assumed that (d) G is not empty.) On P there is given an involution $y=U(x)$, that is, a transformation (1) $y_j=U_j(x)$ with $UU=E$, complex analytic in the following sense. If in E_{2k} the tube T_P with basis P is introduced, that is to say, the point set $x\in P$, $-\infty<\alpha_j<+\infty$ $(j=1,\cdots,k)$, then, for the complex variable $x_j=x_j+i\alpha_j$, $w_j=y_j+i\beta_j$, (1) can be continued into a holomorphic mapping of T_P onto itself.

In T_P is taken a holomorphic function $R(z)$ such that $R(U(z))=(R(z))^{-1}$, $R(z)\neq 0$, $R(x)$ real. For $x\in P$ the following multiple complex integral is set up:

$$S(\mu,\lambda)=\frac{1}{(2\pi i)^k}\int_{x_1-i\infty}^{x_1+i\infty}\cdots\int_{x_k-i\infty}^{x_k+i\infty}\frac{e^{-(\mu,U(z))+(\lambda,z)}}{R(z)}dz_1\cdots dz_k \qquad (\mu\in G).$$

Under general assumptions this integral is absolutely convergent and independent of $x\in P$. Among other results the following are shown. Theorem 1: $S(\mu,\lambda)=0$ for λ non-$\in G$. This generalizes Sonine's integral

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{e^{-\frac{1}{2}\mu+\lambda z}}{z^{\lambda+1}}dz=\begin{cases}\mu^{-\gamma/2}J_\gamma(2(\mu\lambda)^{1/2})\lambda^{\gamma/2}\\ 0\end{cases}$$

$(x<0)$ for the ordinary Bessel function $J_\gamma(x)$. Theorem 2: $p(z_1,\cdots,z_k)$, $q(z_1,\cdots,z_k)$ are polynomials for which $q(U(z))\cdot p(z)=1$. Here $q^*(z)$ is the Lagrange adjoint to $q(z)$. For general $R(z)$, subject to convergence assumptions,

$$q^*\left(\frac{\partial}{\partial\mu_j}\right)\cdot p\left(\frac{\partial}{\partial\lambda_j}\right)S(\mu,\lambda)=S(\mu,\lambda)$$

(generalization of the classical differential equation for ordinary Bessel functions). Theorem 4: The functional transformation $g(\mu)=\int_G S(\mu,\lambda)f(\lambda)dv_\lambda$ is self-inversive.

The following sections deal with unitary transformations and matrix spaces where the general concepts and statements are interpreted in a significant context. *S. C. van Veen* (Delft).

Siegel, Keeve M. An inequality involving Bessel functions of argument nearly equal to their order. Proc. Amer. Math. Soc. 4, 858–859 (1953).

Using a variation of a method of Watson, the author shows that $J_\nu(\nu x)<e^{-\nu F(0,x)}$, where $0<x\leq 1$, $\nu>0$, and

$$F(0,x)=\log\frac{1+(1-x^2)^{1/2}}{x}-(1-x^2)^{1/2}.$$

This improves an inequality of Watson [A treatise on the theory of Bessel functions, 2d ed., Cambridge, 1944, p. 255; these Rev. 6, 64] when $1>(1-x^2)^{1/4}(2\pi\nu)^{1/2}$. *N. D. Kasarinoff* (Lafayette, Ind.).

MacRobert, T. M. An infinite integral involving a product of two modified Bessel functions of the second kind. Proc. Glasgow Math. Assoc. 1, 187–189 (1953).

The integral of the title is $\int_0^\infty x^{l-1}K_m(x)K_n(b/x)dx$, where l, m, and n may be real or complex and $R(b)>0$. The author finds an expansion for it which is the sum of two series of generalized hypergeometric functions (${}_0F_3$'s of argument $b^2/16$). *N. D. Kazarinoff* (Lafayette, Ind.).

Zartarian, G., and Voss, H. M. On the evaluation of the function $f_\lambda(M,\bar{\omega})$. J. Aeronaut. Sci. 20, 781–782 (1953).

The authors expand $\int_0^1 e^{-i\omega u}J_0(\omega u/M)u^\lambda du$ in the form

$$\sum_{n=0}^\infty(-1)^n\left(\frac{a_{2n}(M)}{\lambda+2n+1}-i\omega\frac{b_{2n}(M)}{\lambda+2n+2}\right)\omega^{2n},$$

and tabulate $a_0,\cdots,a_{12}$, $b_0,\cdots,b_{12}$ for $M=5/4$, $10/7$, $3/2$, $5/3$, 2, $5/2$. *A. Erdélyi* (Pasadena, Calif.).

Halphen, Etienne. Une remarquable identité. C. R. Acad. Sci. Paris **237**, 1305–1306 (1953).

The identity

$$\int_0^\infty \exp\left[-s^2\right](\cos xs+\sin xs)s^{-1/2}ds = \int_0^\infty \exp\left[-(s-x/2)^2\right]s^{-1/2}ds$$

results from comparing two integral representations of parabolic cylinder functions. The author gives a proof, due to Morlat, which is based on the comparison of the coefficients of equal powers of x on both sides. *A. Erdélyi.*

van der Held, E. F. M. The contribution of radiation to the conduction of heat. II. Boundary conditions. Appl. Sci. Research A. **4**, 77–99 (1953).

The appendix to this paper contains a summary of the properties of the function $K_n(z)=\int_1^\infty w^{-n}e^{-zw}dw$ (which is closely related to the incomplete gamma function); 4D or 4S tables of K_1, K_2, $2K_3$, $3K_4$, $4K_5$ for

$$z=0(.01).2(.02)2(.5)5(1)8;$$

tables of

$$\int_1^\infty w^{-1}e^{-zw}\log w\,dw,$$

and of various integrals of the form

$$\int_0^\tau K_n(\tau-z)K_2(z)dz,\quad \int_0^\tau K_n(D-\tau+z)K_2(z)dz,$$

$$\int_0^\infty K_n(\tau+z)K_2(z)dz,$$

where n ranges from 1 or 2 to 5, D = 2, 4, and the ranges of τ covered vary. *A. Erdélyi* (Pasadena, Calif.).

Mukherjee, B. N. A note on the second solution of Hermite's equation. Bull. Calcutta Math. Soc. **44** (1952), 124–126 (1953).

W. A. Mersman has obtained a second solution of Hermite's equation

$$\frac{d^2u}{dz^2}-2z\frac{du}{dz}+2nu=0$$

which is linearly independent of $H_n(z)$, viz.,

$$\bar{H}_n(z)=H_n(z)\int_0^z e^{u^2}du+G_n(z)e^{z^2}$$

where $G_n(z)$ is a polynomial of order $(n-1)$

$$G_n(z)=\sum_{k=0}^{(n-1)/2}(-1)^{k+1}\cdot k!\binom{n-k-1}{k}\cdot\sum_{j=0}^{k}\binom{n}{j}(2z)^{n-2k-1};$$

$$G_0(z)=0$$

[J. Math. Physics **29**, 191–197 (1950); these Rev. **13**, 555]. In this paper the author obtains an expression for $G_n(z)$ in terms of $H_n(z)$

$$G_n(z)=\sum_{r=1}^{[(n+1)/2]}(-1)^r2^{r-1}\binom{n-r}{r-1}(r-1)!H_{n-2r+1}.$$

It is shown that $H_n(z)$, $\bar{H}_n(z)$, and $G_n(z)$ satisfy the same functional relation

$$f_{n+1}(z)=-2zf_n(z)+2nf_{n-1}(z)=0$$

from which is deduced

$$H_n(z)=-G_{n+1}(z)-G_n'(z),$$

and

$$H_n(z)G_{n-1}(z)-G_n(z)H_{n-1}(z)=2^{n-1}(n-1)!$$

from which the second expression

$$G_n(z)=-H_n(z)\sum_{r=1}^{n}\frac{2^{r-1}(r-1)!}{H_{r-1}(z)\cdot H_r(z)}$$

results.

The paper ends with the demonstration that

$$\Delta_n(z)=\begin{vmatrix}G_n(z) & G_{n-1}(z)\\ G_{n+1}(z) & G_n(z)\end{vmatrix}>0$$

for all positive integers n and for all real values of z. *S. C. van Veen* (Delft).

Janković, Zlatko. On solutions of Hermite's and Laguerre's differential equation. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 133–148 (1953). (Serbo-Croatian summary)

The author derives the recurrence relation

$$c_{n+1}c_ny_{n-1}-2xc_{n+1}y_n+2(n+1)y_{n+1}=0,$$

where the c_n are arbitrary constants, directly from Hermite's equation and discusses several families of solutions which satisfy the recurrence relation (for $H_n(x)$, $c_n=2n$). This approach yields many classical results. It can be found in Magnus and Oberhettinger, Formeln und Sätze für die speziellen Funktionen . . . [Springer, Berlin, 1948, p. 107; these Rev. **10**, 38] that functions analogous to the author's $h_n(x)$ satisfy the recurrence formula of the $H_n(x)$. Solutions of Laguerre's equation are treated similarly. *N. D. Kasarinoff* (Lafayette, Ind.).

Ragab, F. M. A linear relation between E-functions. Proc. Glasgow Math. Assoc. **1**, 185–186 (1953).

The author proves

$$\begin{aligned}E(\alpha_1,\cdots,\alpha_p:\rho_1,\cdots,\rho_q:x) &= \sum_{r=0}^{2n}(2x)^{-r}C_rE(\tfrac{1}{2}(1+r),\,1+\tfrac{1}{2}r,\,\tfrac{1}{2}(b+1)\\ &\quad -n+r,\,\alpha_1+r,\cdots,\alpha_p+r:\\ &\quad \tfrac{1}{2}(1+b)+r,\,\tfrac{1}{2}-n+r,\,1+r,\,\rho_1+r,\cdots,\rho_q+r:x)\end{aligned}$$

where the C_r are certain coefficients (given in the paper). *A. Erdélyi* (Pasadena, Calif.).

MacRobert, T. M. Some integrals involving E-functions. Proc. Glasgow Math. Assoc. **1**, 190–191 (1953).

The author evaluates several integrals involving his generalized hypergeometric functions. The integrals are

$$\int_0^1\lambda^{\sigma-1}(1-\lambda)^{\rho-\sigma-1}E(\alpha_r:\beta_s:z\lambda^{-m})d\lambda,$$

$$\int_0^\infty f(\lambda)E(\alpha_r:\beta_s:z\lambda^{-m})d\lambda,$$

with $f(\lambda)=e^{-\lambda}\lambda^{\sigma-1}$, $K_n(\lambda^{\frac{1}{2}})\lambda^{\frac{1}{2}k-1}$, or $e^{-\lambda}\lambda^{\sigma-1}E(\gamma,\delta::\lambda)$, and two contour integrals; m is a positive integer. *A. Erdélyi* (Pasadena, Calif.).

Ragab, F. M. Integrals of E-functions expressed in terms of E-functions. Proc. Glasgow Math. Assoc. 1, 192–195 (1953).

The author evaluates the integrals

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_r:\beta_s:\lambda z)d\lambda,\quad \int_0^\infty \lambda^{k-1}E(\alpha_r:\beta_s:\lambda)E(\rho_t:\sigma_u:z/\lambda)d\lambda,$$

$$\int_0^\infty \lambda^{k-1}E(\alpha_r:\beta_s:\lambda)E(\rho_t:\sigma_u:z\lambda)d\lambda$$

under appropriate conditions on the parameters. The particular cases listed in the paper include the Weber-Schafheitlin discontinuous integral. *A. Erdélyi.*

Agostinelli, Cataldo. Nuove funzioni per la risoluzione dei problemi ai limiti relativi al campo ellitico senza fare uso delle transcendenti di Mathieu. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 86, 180–194 (1952).

Let x, y be rectangular coordinates, r, θ polar coordinates, ξ, η elliptic coordinates, so that $x+iy=re^{i\theta}=c\cosh(\xi+i\eta)$. When the wave equation, $\Delta u+\lambda^2u=0$, is solved in a region bounded by an ellipse, it is usual to employ Mathieu functions (which arise when the wave equation is separated in elliptic coordinates). The author proposes instead to employ the functions u_k, u_k^* defined by

$$u_k(\xi,\eta,\lambda)=J_k(\lambda r)\cos k\theta,\quad u_k^*(\xi,\eta,\lambda)=J_k(\lambda r)\sin k\theta.$$

He forms solutions like $U=\sum C_{2m}u_{2m}(\xi,\eta,\lambda)$, and notes that for certain (characteristic) values of λ, the C_{2m} can be determined so that U vanishes on the boundary of the ellipse (for $\xi=\xi_0$).

A number of properties of the functions so determined are proved but it is not at all clear what, if any, advantages these functions have over Mathieu functions.
A. Erdélyi (Pasadena, Calif.).

Schäfke, Friedrich Wilhelm. Das Additionstheorem der Mathieuschen Funktionen. Math. Z. 58, 436–447 (1953).

Wave functions of the elliptic cylinder are solutions of the (two-dimensional) wave equation separated in coordinates based on a confocal family of conics. Such a confocal family is completely characterized by its focal segment, and the wave functions depend on the position and length of that segment. Analytically, such wave functions are represented by products of Mathieu functions and modified Mathieu functions. Addition theorems of Mathieu functions originate from transformation formulas of wave functions of elliptic cylinders belonging to two different focal segments. A product of a Mathieu function and a modified Mathieu function for a given focal segment can be expressed as a sum (in general an infinite sum) of such products formed for a segment whose position, direction, length may all differ from the corresponding quantities for the given segment.

The given wave function, considered as a function on one of the ellipses of the second family, is periodic and hence can be expanded as a series of Mathieu functions belonging to the second focal segment. The integral relationships between various Mathieu functions may be used to show that the coefficients in the expansion are associated Mathieu functions, and the asymptotic behavior of associated Mathieu functions far from the focal segment enables the author to determine the coefficients. If one of the two focal segments degenerates into a point, the Mathieu functions become trigonometric functions, and the associated Mathieu functions become Bessel functions. The analytical proof is based on a number of theorems on Mathieu functions which are taken from a forthcoming book by J. Meixner and the author. *A. Erdélyi* (Pasadena, Calif.).

Bagchi, Hari das, and Mukherji, Bhola Nath. Note on a sequence of functions, defined by a difference-equation. Simon Stevin 29, 185–189 (1952).

The authors obtain an expression for $V=\sum_0^\infty h^nf_n$ when

$$(a_1n+b_1)f_{n+1}+(a_2n+b_2)f_n+(a_3n+b_3)f_{n-1}=0.$$

They envisage in particular the situation that $a_1,\cdots,b_3$, and hence f_n, are analytic functions of a complex variable z.
A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Fichera, Gaetano. A proposito delle mie Note "Sui teoremi di esistenza della teoria del potenziale e della rappresentazione conforme". Boll. Un. Mat. Ital. (3) 8, 109–114 (1953).

Observations on an earlier paper by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 356–360 (1951)] and its review [these Rev. 13, 931].
Z. Nehari (St. Louis, Mo.).

Walsh, J. L. Note on the shape of level curves of Green's function. Amer. Math. Monthly 60, 671–674 (1953).

ψ désignant l'angle du rayon vecteur Om avec la tangente en m à une courbe, l'auteur donne diverses majorations de $|\psi-\pi/2|$ sur la courbe $G=-\log r$, G désignant la fonction de Green de pole O d'un domaine R contenant O. Cas où le domaine est illimité, G ayant son pole à l'infini.
J. Lelong (Lille).

Myškis, A. D. On an extremal property of the solution of the first boundary problem in potential theory. Izvestiya Akad. Nauk SSSR. Ser. Mat. 17, 13–30 (1953). (Russian)

Further properties are developed for the so-called "load function" introduced by the author [in Vestnik Moskov Univ. Ser. Fiz.-Mat. Estest. Nauk 1946, 131–136 (unavailable for review)]. This is defined in the following way. Let G be a region of n-dimensional space and Γ its boundary (for simplicity we discuss only the case of $n=2$). Given a C' real-valued function u on G and a simple piecewise smooth closed path S in G, we set

$$\Pi(u,S)=\int_S\frac{\partial u}{\partial n}dS\quad(n=\text{inward normal}).$$

The load function H then appears as

$$(*)\qquad H(u)=\sup\sum_{k=1}^n|\Pi(u,S_k)|,$$

where the sup is taken over all finite collections of paths $S_1, S_2,\cdots,S_n$ whose interior regions are pairwise disjoint (the "interior" is here determined by the orientation of S rather than by boundedness considerations).

The main result is an extremal property of H, to wit: let u be a continuous function on $\bar G$ admitting on G a continuous Privaloff operator; if v is the solution of the generalized Dirichlet problem for the boundary function $u|\Gamma$, then $H(v)\leqq H(u)$. Further, when $v\not\equiv u$ and Γ has a finite number of components, then $H(v)<H(u)$.

In the opinion of the reviewer the theory would be improved by requiring the paths $S_1, S_2, \cdots, S_n$ to have their interior regions contained in G (rectangular paths would suffice). The contribution in (*) corresponding to the boundary components would then be eliminated. This contribution seems merely to obscure the situation, but it could be treated separately if desired. With the resulting modification in the definition of $H(u)$ it can be shown that $H(u)$ is finite if and only if u is an almost δ-subharmonic function whose mass distribution has finite total variation. In fact, the total variation is just $(1/2\pi)H(u)$. The extremal property of H is now obvious from the observation that $H(u)$ is zero for u harmonic but is positive for all other admissible u. Moreover, the hypothesis that u admit a continuous Privaloff operator serves only to ensure δ-subharmonicity of u and can therefore be discarded (for $u \in C'$), along with the requirement that the boundary components be finite in number. *M. G. Arsove* (Seattle, Wash.).

Komatu, Yûsaku, and Hong, Imsik. On mixed boundary value problems. Kōdai Math. Sem. Rep. **1953**, 65–76 (1953).

Treatment, from a different point of view, of the mixed boundary-value problem studied by the first author in a previous paper [J. Fac. Sci. Univ. Tokyo. Sect. I. **6**, 345–391 (1953); these Rev. **14**, 969]. *Z. Nehari.*

Aržanyh, I. S. Vector potentials of the dynamics of an elastic body. Doklady Akad. Nauk SSSR (N.S.) **88**, 961–964 (1953). (Russian)

In the present paper the author outlines the construction of a theory of vector potentials for the vector differential operator

$$\Delta^* \equiv \alpha \operatorname{grad} \operatorname{div} - \beta \operatorname{rot} \operatorname{rot} - \eta^2, \quad \eta = \text{const.}, \quad \alpha = \frac{\lambda + 2\mu}{\rho}, \quad \beta = \frac{\mu}{\rho},$$

which occurs in the dynamical theory of an isotropic elastic body. This construction is based on a new set of fundamental solutions, which differ from those employed by Kupradze [Boundary problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; these Rev. **15**, 318] who employs Stokes' fundamental solutions [see Love, Mathematical theory of elasticity, 4th ed., Cambridge, 1927]. In the limit, as $\eta \to 0$, both the fundamental solutions employed by the author, as well as Stokes' fundamental solutions, approach Kelvin's fundamental solutions (see Love's book). *J. B. Diaz.*

Grinberg, G. A., Lebedev, N. N., and Uflyand, Ya. S. A method of solution of a general biharmonic problem for a rectangular region with given values of the function and its normal derivative on the contour. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 73–86 (1953). (Russian)

The boundary value problem in question is the determination of $w(x, y)$ such that

$$\Delta\Delta w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = Q(x, y),$$

in a domain D, and $w = f(s)$, $\partial w/\partial \nu = g(s)$, on the boundary of D, where ν denotes the outer normal and Q, f, g are given functions. Section 1 describes the general method of approximation, which consists essentially in determining the harmonic function $\Delta w - \Delta w_0$ (where w_0 is any chosen particular solution of $\Delta\Delta w_0 = Q(x, y)$ in D) as a linear combination $\sum_{n=0}^{\infty} a_n \psi_n$ of an orthonormal ($\iint_D \psi_m \psi_n dxdy = 0$ if $m \neq n$, $= 1$ if $m = n$) complete sequence of functions ψ_n, which are harmonic in D [see Grinberg, Doklady Akad. Nauk SSSR (N.S.) **76**, 661–664 (1951); these Rev. **13**, 184; this method is similar to that proposed by Miranda, Rend. Sem. Mat. Univ. Roma (4) **1**, 262–266 (1937)]. Given an orthonormal sequence of harmonic functions for D, numerical approximations for Δw, which is of direct interest in the theory of plates, may be calculated. If approximations to w are desired, a knowledge of Δw, plus the boundary conditions above, enable one to determine w by solving either a Dirichlet or a Neumann problem. Section 2 contains the explicit construction and tables of a suitable orthonormal sequence of harmonic functions for a rectangular domain. Sections 3 and 4 contain applications to various problems of a square plate under symmetrically placed concentrated loads and to the plane problem of the theory of elasticity. *J. B. Diaz* (College Park, Md.).

Nikol'skiĭ, S. M. Second note on the continuation of differentiable functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **88**, 17–19 (1953). (Russian)

Nikol'skiĭ, S. M. Properties of differentiable functions of several variables on closed smooth manifolds. Doklady Akad. Nauk SSSR (N.S.) **88**, 213–216 (1953). (Russian)

Discussion of the behavior of functions in the author's [Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); these Rev. **14**, 32] classes $H_p^{(r_1, \cdots, r_n)}(G; M_1, \cdots, M_n)$ with regard to the properties mentioned in the titles. The principal results are too involved to be summarized here. *M. G. Arsove.*

Nikol'skiĭ, S. M. On the solution of the polyharmonic equation by a variational method. Doklady Akad. Nauk SSSR (N.S.) **88**, 409–411 (1953). (Russian)

For a given region Ω and under appropriate boundary conditions the solution of the n-dimensional polyharmonic equation $\Delta^\nu u = 0$ appears as the function minimizing the integral

$$D_\nu(f) = \int_\Omega \sum_{\sum \alpha_i = \nu} \frac{\nu!}{\alpha_1! \cdots \alpha_n!} \left(\frac{\partial^\nu f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right)^2 d\Omega$$

with respect to the class of all functions f satisfying the given boundary conditions and making $D_\nu(f)$ finite. The author derives a rather complicated condition sufficient to ensure that this class is not empty. *M. G. Arsove.*

McMahon, James. Lower bounds for the electrostatic capacity of a cube. Proc. Roy. Irish Acad. Sect. A. **55**, 133–167 (1953).

A systematic method is developed in order to obtain lower bounds for the capacity C of a solid bounded by a surface B. The following inequality is a consequence of the Kelvin-Trefftz principle:

$$4\pi C > 4\pi a_0 - a_0 e_0 - \sum_{\nu=1}^{n} a_\nu e_\nu,$$

where a_0, a_ν are arbitrary and

$$e_0 = a_0 \int_V (\nabla r^{-1})^2 dV + \sum_{\nu=1}^{n} a_\nu \int_V \nabla r^{-1} \cdot \mathbf{p}_\nu dV - 4\pi,$$

$$e_\nu = a_0 \int_B r^{-1} \mathbf{p}_\nu \cdot \mathbf{n} dB + \sum_{\rho=1}^{n} a_\rho \int_V \mathbf{p}_\nu \cdot \mathbf{p}_\rho dV;$$

here $\mathbf{p}_\nu$ are arbitrary source-free vectors defined in the exterior space V having the order r^{-2} at infinity. Also

$\int_B \mathbf{p}_r \cdot \mathbf{n} dB = 0$. We seek a set a_0, a_r for which $e_0 = e_r = 0$ holds at least approximately.

The vectors $\mathbf{p}_r$ are defined as follows. We divide V into tetrahedra. Each point r is surrounded by a "cell" composed of all tetrahedra having a common vertex at r. To each r we define a scalar function P_r which equals 1 at r, linear within the cell and 0 outside and on the surface surrounding the cell. We define then the vector $\mathbf{p}_{rs} = \text{grad } P_r \times \text{grad } P_s$ and $\mathbf{p}_r = \mathbf{p}_{rr}$. In case of a cube of edge 1, a finite part of the space V is divided into lattice cubes having as edge $1/m$, m an integer. We insert points at the centre of each cube and at the centre of each face. Joining these centres with the corners and with each other, three types of cells are obtained: (1) the lattice cube itself surrounding its centre and consisting of 24 tetrahedra; (2) an octahedron surrounding the centre of a face and consisting of 8 tetrahedra; (3) a rhombic dodecahedron surrounding a corner and consisting of 48 tetrahedra. The resulting linear equations are partly simplified by using the symmetry of the cube. The inequality $C > 0.639273$ obtained by the author is better than the bound $C > 0.62033$ obtained by Pólya-Szegö [Isoperimetric inequalities in mathematical physics, Princeton, 1951, pp. 76–78; these Rev. 13, 270]. *G. Szegö.*

Differential Equations

Sumner, D. B. A differential system of infinite order with non-vanishing solutions. Proc. London Math. Soc. (3) 3, 464–479 (1953).

Let $E(w) = \prod_1^\infty (1 - w^2/\lambda_n^2)$, $\lambda_n = \rho + n - \Delta + 2\Delta\alpha_n$, $\rho \geq 0$, $0 \leq \alpha_n \leq 1$, $0 < \Delta < \frac{1}{4}$, and consider the infinite order differential equation and boundary conditions (*) $DE(D)f(x) = 0$, $f(\pm\infty) = 0$, $f \in A$, where A is the class of analytic functions $\{f(z)\}$ for which $\sum_0^\infty [(iy)^n/n!] f^{(n)}(x)$ converges uniformly on the strip $|y| \leq \theta < \pi$, x arbitrary (for each θ). There exists a function $k(y) \in L^p[-\infty, \infty]$ for a $p \geq 2$ such that (*) is equivalent to (**) $\lim_{\theta \to \pi} \int_{-\theta}^{\theta} k(y) f'(x+iy) dy = 0$, $f(\pm\infty) = 0$, $f \in A$. It is then shown that there is a function $h(x)$, which together with all its even derivatives, satisfies (**) and therefore (*). Other solutions of this kind are obtained as follows. Let $F(w)$ be an even function, analytic in every sector $|\arg w| \leq \alpha < \pi/2$, with $|F(w)| < A \exp(\epsilon R)$ for arbitrary small ϵ and sufficiently large R ($|w| \leq R$). Set

$$J_\beta(z) = \frac{1}{2\pi i} \int_0^{\infty(\arg w = \beta)} \frac{\exp(-zw) F(w)}{E(w)} dw.$$

This integral has a value $J_1(z)$, the same for all positive β, and a value $J_2(z)$ for negative β. Set $H(z) = J_1(z) - J_2(z)$. Then $H(z)$ is analytic in the z-plane cut along $(-i\infty, -i\pi)$, $(i\pi, i\infty)$, and $H \in A$. Moreover, $H(x)$ and all its even derivatives satisfy (**). The above results parallel earlier work of Bochner and Widder on a simpler equation of form (*) [Bull. Amer. Math. Soc. 54, 409–415 (1948); these Rev. 9, 437]. *I. M. Sheffer* (State College, Pa.).

Walmsley, Charles. Null trigonometric series in differential equations. Canadian J. Math. 5, 536–543 (1953).

Let $F(D)$ be a polynomial of degree m in the differential operator D with constant coefficients and let $f(x) \sim \sum \gamma_n e^{inx}$ be uniformly summable (C, k), k being a positive integer, in $(-\pi, \pi)$. Then the general solution of the ordinary differential equation $F(D)y = f(x)$ is given by $y \sim \sum c_n e^{inx}$, which is convergent if $k \leq m$ and uniformly summable $(C, k-m)$ if $k \geq m$. The Fourier coefficients c_n are given by

$$c_n = \left[\gamma_n + (-1)^n \sum_{r=1}^m A_r n^{r-1} \right] \Big/ F(in),$$

if $F(in) \neq 0$, otherwise c_n are arbitrary, where A_r $(r = 1, \cdots, m)$ are arbitrary constants, but $\gamma_n + (-1)^n \sum_{r=1}^m A_r n^{r-1} = 0$ for n such that $F(in) = 0$, if any. Several examples of this theorem are given. The proof depends on the theory of trigonometrical series of S. Verblunsky [Proc. London Math. Soc. (2) 34, 457–491 (1932)] and F. Wolf [ibid. 45, 328–356 (1939); these Rev. 1, 225]. *S. Izumi* (Tokyo).

Wintner, Aurel. On a theorem of Bôcher in the theory of ordinary linear differential equations. Amer. J. Math. 76, 183–190 (1954).

The author considers the complex, linear homogeneous differential system $x' = A(t)x$, where the $n \times n$ matrix $A(t)$ is continuous for large positive t. Then $A(t)$ is said to have property (*) (studied by Bôcher) in case $x(\infty) = \lim_{t\to\infty} x(t)$ exists for every solution vector $x(t)$. Define $A(t) \in R$ in case $\lim_{T\to\infty} \int^T A(t) dt$ exists and then write

$$A^0(t) = \lim_{T\to\infty} \int_t^T A(t) dt.$$

Theorem: If $A(t) \in R$ and if either $A^0(t)A(t)$ or $A(t)A^0(t)$ is absolutely integrable on a right half-line $c \leq t < \infty$, then $A(t)$ has (*). This theorem is stronger than several known sufficiency conditions that $A(t)$ have (*); for example, that $A(t)$ is absolutely integrable on a right half-line. The author then gives two examples to show that the single condition $A(t) \in R$ is neither necessary nor sufficient that $A(t)$ have (*). The second from bottom line on p. 187 contains a misprint and should read, ". . . that $U(t) \in L(\subset R)$ if . . . " *L. Markus* (New Haven, Conn.).

Erugin, N. P. The methods of A. M. Lyapunov and questions of stability in the large. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 389–400 (1953). (Russian)

This is a general report on problems related to the asymptotic stability in the large of a system

$$(1) \qquad \dot{x} = Px + f(x), \quad f(x) = O(\|x\|),$$

where x, f are n-vectors and P is a constant matrix. 1. Let A be the set of all initial points x^0 such that the solution starting there approaches 0 as $t \to +\infty$. The set A is open and its boundary consists of complete motions. If the characteristic roots λ_j of P have negative real parts the solutions near the origin are representable as power series in x^0 and the $\exp \lambda_j t$. They converge in a certain region $B \subset A$. When is $B = A$? Under what conditions is A the whole space of x? Only partial results are known. 2. Relation to the approximation methods of van der Pol and Krylov-Bogolyubov for systems $\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x}, t)$. 3. Existence and properties of a Lyapunov function in the large. Open question: does there exist such a function even if the "asymptotic domain" A is the whole space? (Probably not.) For a restricted A the existence of a Lyapunov function has been proved by Massera. Various detailed special results for $n = 2$ are also discussed.

[References: Erugin, same journal 14, 459–512 (1950); 15, 227–236 (1951); 16, 620–628 (1952); these Rev. 12, 412, 705; 14, 376; Bogolyubov, On some statistical methods in mathematical physics, Izdat. Akad. Nauk USSR, Lvov,

1945; these Rev. 8, 37; Krylov and Bogolyubov, Introduction à la mécanique non-linéaire, ibid., Kiev, 1937; Goršin, Izvestiya Akad. Nauk Kazah. SSR **1948**, no. 56, Ser. Mat. Meh. **2**, 74–101; these Rev. **14**, 48; Barbašin and Krasovskiĭ, Doklady Akad. Nauk SSSR (N.S.) **86**, 453–456 (1952); these Rev. **14**, 646; Eršov, Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 61–72 (1953); these Rev. **14**, 752; Krasovskiĭ, Doklady Akad. Nauk SSSR (N.S.) **88**, 401–404 (1953); Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 339–350 (1953); these Rev. **14**, 752, 1087; Malkin, ibid. **16**, 365–368 (1953); these Rev. **14**, 48; Stebakov, Doklady Akad. Nauk SSSR (N.S.) **82**, 677–680 (1952); these Rev. **14**, 274.]

S. Lefschetz (Princeton, N. J.).

Latiševa, K. Ya. On asymptotic solutions of linear differential equations in the case of a double root of the characteristic equation. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 14–19 (1951). (Ukrainian)

Assume that the coefficients in the differential equation (1) $y^{(n)}+\sum_{j=1}^{n}p_j(x)y^{(n-j)}=0$ have the (convergent or asymptotic) expansions $p_j(x)=x^{jk}(p_{j0}+p_{j1}x^{-1}+\cdots)$ in decreasing powers of x, where the rank k is an integer. Then to distinct zeros of the characteristic polynomial $\alpha^n+\sum_{j=1}^{n}p_{j0}\alpha^{n-j}$ belong solutions asymptotically represented by normal series $e^{Q(x)}x^{\rho}(c_0+c_1x^{-1}+\cdots)$, where $Q(x)$ is a polynomial in x^{-1}. The author establishes conditions under which there exist two normal solutions belonging to a double zero of the characteristic polynomial.

M. Golomb.

Smith, R. A. On the singularities in the complex plane of the solutions of $y''+y'f(y)+g(y)=P(x)$. Proc. London Math. Soc. (3) **3**, 498–512 (1953).

This paper is concerned with the study of the behavior in the complex plane of the solutions of

$$(*)\qquad y''+y'f(y)+g(y)=p(x),$$

where f and g are polynomials of degree n and m $(n>m)$, respectively. Applying classical methods going back to Painlevé and Malmquist, the author proves that (*) has an infinite family of solutions of the form $y=\sum_{r=-1}^{\infty}a_r(x-x_0)^{r/n}$ $(a_{-1}\neq 0)$ in the neighborhood of a point x_0 at which $p(x)$ is regular. He then shows that if a solution of (*) can be continued analytically along a continuous curve γ up to, but not beyond, a point x_0, the latter is either an algebraic critical point of the foregoing type, or else a limit point of such points; if γ is a contour, the second case is excluded. Finally, it is shown by means of an example that a solution of (*) may indeed have a singularity of the second type.

Z. Nehari (St. Louis, Mo.).

Colombo, Giuseppe. Sulle oscillazioni forzate di un circuito comprendente una bobina a nucleo di ferro. Rend. Sem. Mat. Univ. Padova **22**, 380–398 (1953).

A non-linear L, R, C circuit containing an iron core reactor is discussed relative to the existence of a certain stable subharmonic of the exciting frequency. In particular, the equation

$$(1)\qquad f'(\dot q)\ddot q+R\dot q+C^{-1}q=E(t)$$

is considered, where the magnetic flux $f(\dot q)$ is "almost" piecewise linear, $E(t)$ is "almost" piecewise constant and of period $2T$. It is then shown if in addition, (i) R is small, (ii) $f'(\dot q)$ is "almost" zero for $|\dot q|\geqq I$ and I is sufficiently small, (iii) $\pi/2<T<5\pi/6$, that (1) possesses a stable subharmonic solution with minimum period $6T$. (The periodic solution of period $2T$ is not stable.) This is accomplished by first showing that the approximate equation

$$(1')\qquad f_0'(\dot q)\ddot q+C^{-1}q=E_0(t),$$

with $f_0'(\dot q)=L_1$ or L_2 (constants) for $|\dot q|\geqq I$ or $|\dot q|<I$ respectively, $L_1\ll L_2$, $I\ll 1$ and $E_0(t)=-E^*$ or E^* (constant) according as $0\leqq t<T$ or $T\leqq t<2T$ respectively, has the required stable subharmonic solution. Then by a simple application of the Brouwer fix-point theorem the same conclusion holds for (1). Along with this case, the author discusses in parallel, and much more briefly, the case where R and T are sufficiently large, but the other smallness conditions continue to remain valid. Then a stable periodic solution of period $2T$ exists which practically coincides with $E(t)$.

C. R. DePrima (Pasadena, Calif.).

Cahen, Gilbert. Systèmes électromécaniques non linéaires. Rev. Gén. Electricité **62**, 277–293 (1953).

An extension of the graphical method of Lienard is given which enables one to discuss systems governed by differential equations of the two types

$$\ddot x+f(\dot x)+\dot x b(x)+r(x)=0,\quad \ddot x+\dot x^2a(x)+\dot x b(x)+c(x)=0.$$

A topological discussion of the behaviour of the solution curves near singularities is provided. In the second portion of the paper, the method is applied to the study of a complicated closed-loop system composed of a non-linear oscillator and a low-pass filter. Such questions as entrainment, threshold of synchronization and the synchronization of coupled oscillators are considered.

H. Cohen.

Obi, Chike. Researches on the equation

$$(\mathrm{E})\qquad \ddot x+(\epsilon_1+\epsilon_2x)\dot x+x+\epsilon_3x^2=0.$$

Proc. Cambridge Philos. Soc. **50**, 26–32 (1954).

The author treats equation (E) of the title in the case ϵ_2/ϵ_3 small and adopts the standard form

$$(\mathrm{E}^*)\qquad \ddot x+(k+\epsilon x)\dot x+x+\tfrac{2}{3}x^2=0$$

obtained from (E) by a change of variables. Using the perturbation method of Poincaré with the family of periodic solutions of the equation $\ddot x+x+\frac{2}{3}x^2=0$ as generating solutions, the author obtains the following results: (1) A necessary condition for (E*), with ϵ small, to have a periodic solution is that $k=\epsilon\mu$, where $0<\mu\leqq 2/21$; (2) when $k=\epsilon\mu$ and $0<\mu<2/21$, then (E*) has (save for translations in t) a unique periodic solution.

C. E. Langenhop.

Minorsky, N. On interaction of non-linear oscillations. J. Franklin Inst. **256**, 147–165 (1953).

The use of words in the title of this paper is peculiar: what is here termed interaction of oscillations would ordinarily be termed dependence of an oscillation on a parameter. The author first reviews various known facts concerning the existence and stability of oscillations of non-linear systems. Then he applies the stroboscopic method to a study of the periodic solutions of the equation

$$\ddot x+e(x^2-1)\dot x+[1+(a-cx^2)\cos 2t]x=0.$$

As would be expected, the solutions display affinities with the solutions of the van der Pol equation, and with those of the Mathieu equation. Attention is restricted to the case in which the constants a, c, e are small, and the results obtained are not regarded as representing more than a first approximation.

L. A. MacColl (New York, N. Y.).

Hayashi, Chihiro. Forced oscillations with nonlinear restoring force. Mem. Fac. Eng. Kyoto Univ. **13**, 180–197 (1951).

Transient solutions of $d^2v/d\tau^2+kdv/d\tau+v^3=B\cos n\tau$, $n=1$ or $n=3$, are studied by the method of van der Pol [Philos. Mag. (7) **3**, 65–80 (1927)]. Graphs are given showing which initial conditions lead to which periodic solution. These are compared with experimental graphs obtained from an electric circuit with an iron core inductance. The experimental curves exhibit an interesting branching pattern. No proofs are given that the behavior of the approximate solutions mirrors the behavior of the true solutions.
F. M. Stewart (Providence, R. I.).

Hayashi, Chihiro. Forced oscillations with nonlinear restoring force. J. Appl. Phys. **24**, 198–207 (1953).

A reprint of the paper reviewed above.
F. M. Stewart (Providence, R. I.).

— ***Wasow, W. Sur les problèmes de perturbation singuliers dans la théorie des vibrations non linéaires. On singular perturbation problems in the theory of non-linear vibrations.** Actes du Colloque International des Vibrations non linéaires, Ile de Porquerolles, 1951, pp. 207–222; discussion, p. 219, Publ. Sci. Tech. Ministère de l'Air, Paris, no. 281 (1953). (French and English)

An expository account of results of Friedrichs and Wasow, Levinson, Tihonov, Volk and Wasow on periodic solutions of singular perturbation problems. *N. Levinson.*

Kononenko, V. O. On a case of coupled oscillations. Dopovidi Akad. Nauk RSR **1951**, 74–81 (1951). (Ukrainian. Russian summary)

Application of the approximation methods of Krylov-Bogolyubov to a pair of differential equations of quasi-harmonic type. It is shown in particular that certain coordinates seemingly non-coupled under linear approximation are actually coupled. *S. Lefschetz.*

Mitropol'skiĭ, Yu. A. On oscillations in gyroscopic systems while passing through resonance. Ukrain. Mat. Žurnal **5**, 333–349 (1953). (Russian)

Mathematically speaking what is accomplished is as follows. One considers a system:

$$a(\tau)\ddot{x}+\gamma(\tau)\dot{y}+b(\tau)x=\epsilon f_1(\tau,\theta,x,\dot{x},y,\dot{y}),$$
$$c(\tau)\ddot{y}-\gamma(\tau)\dot{x}+d(\tau)x=\epsilon f_2(\cdots);$$

$\tau=\epsilon t$, $\dot{\theta}=\nu(\tau)$, f_1 and f_2 have periods 2π in θ. Upon considering $a,\cdots,d$ as constant and making $\epsilon=0$ at the right, one has a solution with two periods $\omega_1(\tau)$, $\omega_2(\tau)$. Assuming that, for some τ^*, $\omega_1/\omega_2=s/r$, where s, r are relatively prime small positive integers, the author deals with the system by the appropriate approximation methods of Krylov-Bogolyubov. [References: Krylov and Bogolyubov, Introduction to non-linear mechanics, Princeton, 1943; these Rev. **4**, 142; Mitropol'skiĭ, Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 139–170 (1950); these Rev. **12**, 181.] *S. Lefschetz.*

Freud, G. Über die Mohrensteinsche Berechnung des H_2 Moleküls. Acta Phys. Acad. Sci. Hungar. **1**, 325–328 (1952). (Russian summary)

If $y(x)=\sum_{\nu=0}^{\infty}a_\nu x^\nu$ is a solution of

$$[(x^2-1)y']'+\left(\sum_{k=0}^{m}c_kx^k\right)y=0,$$

then $\alpha_0=\lim_{n\to\infty}2na_{2n}$ and $\alpha_1=\lim_{n\to\infty}(2n-1)a_{2n-1}$ exist. Also

$$y(x)=\tfrac{1}{2}\{(\alpha_1-\alpha_0)\log(1+x)-(\alpha_1+\alpha_0)\log(1-x)\}+\sum_{\nu=0}^{\infty}b_\nu x^\nu,$$

where $|b_\nu|<N/\nu^{3/2}$. *F. M. Stewart* (Providence, R. I.).

Volpato, Mario. Sopra un problema al contorno per l'equazione differenziale $y^{(n)}=f(x,y,y',\cdots,y^{(n-1)})$. Rend. Sem. Mat. Univ. Padova **22**, 334–349 (1953).

This paper gives a sufficient condition for the existence of a solution of the system comprising the equation in the title and the boundary conditions $y(x_j)=c_j$, $j=1,\cdots,n$. The theorem implies those of Cinquini [Ann. Scuola Norm. Super. Pisa (2) **9**, 61–77 (1940); these Rev. **2**, 51] and Zwirner [these Rev. **8**, 206]. The hypothesis of the theorem, too long to state here, involves the following: if G is the polynomial which has degree at most $n-1$ and satisfies the boundary conditions, then each absolutely continuous y which approximately equals G and satisfies $|y^{(n)}|\leq|f|$ must also satisfy $|f|\leq F$, where F involves only the two variables x, $y^{(n-1)}$ and a positive bound M for the coefficient $|G^{(n-1)}|$. *J. M. Thomas* (Durham, N. C.).

Raševskiĭ, P. K. On the extension of the operational calculus to boundary problems. Uspehi Matem. Nauk (N.S.) **8**, no. 4(56), 65–80 (1953). (Russian)

It is well known that in the application of the Heaviside calculus to ordinary differential equations, boundary conditions may be incorporated adroitly by the use of delta functions and their derivatives. The author develops this idea.

First, he considers the differential equation $(D+a)\chi=\phi$ where a is a constant, $D=d/dx$, and ϕ (and hence χ) is a generalized function, that is, a combination of a proper function and delta functions and their derivatives. He then passes to an equation of higher order (with constant coefficients) and to systems of ordinary linear equations. When D is the operator d^2/dx^2, two-point boundary conditions are introduced, and consequently the delta functions and their derivatives have two singular points.

To some extent the method can be carried over to systems of linear partial differential equations where D is replaced by Laplace's operator, and Dirac's delta function by the delta function appropriate to the boundary of the region in which the differential equations are studied. *A. Erdélyi.*

Urabe, Minoru. Certain singularity of ordinary differential equations of three variables. J. Sci. Hiroshima Univ. Ser. A. **16**, 57–60 (1952).

The system is $P^{-1}dx=Q^{-1}dy=R^{-1}dz$ with P, Q, R holomorphic in x, y, z and vanishing at (0, 0, 0). The characteristic roots of the matrix of the coefficients of the linear terms in P, Q, R are assumed to form a triangle whose interior holds 0. The shape of the integral varieties in the neighborhood of the origin is studied. *J. M. Thomas.*

Matsushima, Yozô. On a theorem concerning the prolongation of a differential system. Nagoya Math. J. **6**, 1–16 (1953).

E. Cartan proved that the prolonged system of a differential system in involution is in involution, when the given system consists of equations of degrees 0, 1, 2 [Ann. Sci. Ecole Norm. Sup. (3) **21**, 153–206 (1904)]. The author gives a detailed proof of this theorem for a general differential system. *S. Chern* (Chicago, Ill.).

Birindelli, Carlo. Estensioni al teorema di Meyer, sulle soluzioni di sistemi di equazioni ai differenziali totali, di un perfezionamento costruttivo del Picone e qualche ulteriore osservazione. Univ. Roma. Ist. Alta Mat. Rend. Mat. e Appl. (5) **12**, 177–187 (1953) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **378** (1953).

Let the system of total differential equations

$$dx_i=\sum_{j=1}^{r} f_{ij}(t_1, \cdots, t_r, x_1, \cdots, x_p)dt_j \quad (i=1, \cdots, p),$$

where the f_{ij} satisfy certain simple regularity conditions, be completely integrable. A classical theorem asserts that there exists a unique solution, defined over a rectangular domain D: $|t_j-t_j^0|<a_j$ $(j=1, \cdots, r)$, such that the functions x_i assume given values x_i^0 at the point $(t_1^0, \cdots, t_r^0)$. This paper is concerned with refinements of the proof, designed to lead to a rectangular domain of existence which is as large as possible. The results of the discussion involve much complicated detail in an essential way, and hence they cannot be summarized briefly. *L. A. MacColl.*

HAVE

***Cimmino, Gianfranco. Problemi di valori al contorno per alcuni sistemi di equazioni lineari alle derivate parziali.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 61–65. Casa Editrice Perrella, Roma, 1953.

The author previously has studied the solutions and compatibility conditions of systems of linear partial differential equations [Rivista Mat. Univ. Parma **1**, 105–116 (1950); these Rev. **12**, 503]. He now applies his general heuristic procedure to concrete special cases, for example to $X_x-Y_y=f$, $X_y+Y_x=g$, where X, Y are unknown functions of x, y. *E. F. Beckenbach* (Los Angeles, Calif.).

Hornich, Hans. Die Existenz von regulären Lösungen bei allgemeinen linearen partiellen Differentialgleichungen. Rend. Circ. Mat. Palermo (2) **2**, 46–52 (1953).

The question treated is: does the equation

$$\sum g(i_1, \cdots, i_n)u(i_1, \cdots, i_n)=f,$$

where the single f and the finite set of g's are given power series, u is an unknown power series in independent variables $x_1, \cdots, x_n$ and the derivative of u with respect to monomial $x_1^{i_1}\cdots x_n^{i_n}$ is $u(i_1, \cdots, i_n)$ have a solution? There are given necessary conditions (i) that coefficients of u exist and (ii) that the corresponding series u converge for all f. *J. M. Thomas* (Durham, N. C.).

Kasuga, Takashi. On the linear partial differential equation of the first order. Osaka Math. J. **5**, 211–225 (1953).

The equation is $p+f(x, y, z)q=g(x, y, z)$ with f_y, g_y, f_z, g_z continuous in open D with projection G on $z=0$. The following is proved: if surface S has continuous z satisfying the equation almost everywhere in G, each curve $dy/dx=f$, $dz/dx=g$ which is in D and meets S lies in S; if in addition z_y exists everywhere in G, then z has a total differential and satisfies the equation everywhere in G. *J. M. Thomas* (Durham, N. C.).

Neišuler, L. Ya. On a three-term separation of the variables in an equation with four variables. Doklady Akad. Nauk SSSR (N.S.) **82**, 189–192 (1952). (Russian)

E. Goursat [Bull. Soc. Math. France **27**, 1–34 (1899)] gave the second-order partial differential equation $F_1F_{23}=F_2F_{13}$ as a necessary and sufficient condition for the relation $F(x_1, x_2, x_3, x_4)=0$ to be equivalent to $x_4=f(g(x_1, x_2), x_3)$. In the present paper necessary and sufficient conditions for the stated relation to be equivalent to

$$x_4=f(g(x_1, x_2), h(x_1, x_3))$$

are shown to be the fourth-order equations obtained by setting $i, j, k=2, 3, 3$; $3, 2, 2$; $2, 3, 4$; $3, 2, 4$, respectively, in

$$\left\{\left[\frac{F_j}{F_i}\left(\frac{F_1}{F_j}\right)_i\right]_j\Big/\left[\frac{F_j}{F_i}\left(\frac{F_i}{F_j}\right)_i\right]_j\right\}_k=0.$$

R. Church (Monterey, Calif.).

HAVE

***Görtler, Henry. Über nicht-lineare partielle Differentialgleichungen vom Reibungsschicht-Typus.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 116–120. Casa Editrice Perrella, Roma, 1953.

After expository remarks concerning several procedures in boundary-layer calculations, the author discusses briefly a comparison theorem which should be useful in making error estimates for approximate boundary-layer solutions, or in studying the dependence of solutions on parameters and boundary conditions. [For this comparison theorem, see Görtler, Z. Angew. Math. Mech. **30**, 265–267 (1950).] *D. Gilbarg* (Bloomington, Ind.).

Goldstein, S. On the mathematics of exchange processes in fixed columns. I. Mathematical solutions and asymptotic expansions. Proc. Roy. Soc. London. Ser. A. **219**, 151–171 (1953).

This paper and the one following represent a consolidation and coordination of work done by many investigators on the analysis of one-dimensional exchange processes. The general problem is the solution of the simultaneous equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \frac{\partial v}{\partial y}=u-rv+(r-1)uv$$

with boundary conditions $v=0$ at $y=0$ for $x\geqq 0$ and $u=u_0$ at $s=0$ for $y>0$. This is solved as $u=N/F$ and $v=M/F$ where for constant u_0:

$$F=e^{(1-r)(x-y)}[1-J(x, ry)]+e^{rx(1/\beta-1)}e^{-(1-\beta)y}J(rx/\beta, \beta y),$$

$$N=u_0e^{rx(1/\beta-1)}e^{-(1-\beta)y}J(rx/\beta, \beta y),$$

$$M=\frac{u_0}{\beta}e^{rx(1/\beta-1)}e^{-(1-\beta)y}[1-J(\beta y, rx/\beta)],$$

where $1-\beta=(1-r)(1-u_0)$; as obtained by operational procedures. The function

$$J(x, y)=e^{-(x+y)}\sum_{m=0}^{\infty}(y/x)^{m/2}I_m(2(xy)^{\frac{1}{2}})$$

is examined with respect to its integral representation and asymptotic expansions as well as repeated derivatives and Taylor series. Extensive reference and acknowledgment is made to other contributors to the problem. *N. A. Hall.*

Goldstein, S. On the mathematics of exchange processes in fixed columns. II. The equilibrium theory as the limit of the kinetic theory. Proc. Roy. Soc. London. Ser. A. **219**, 171–185 (1953).

Extending the analysis of the preceding paper, the limiting case of so-called equilibrium exchange is considered. In this case the second of the simultaneous partial differential equations reduces to $u-rv+(r-1)vu=0$ so that explicit solutions for a number of combinations of boundary conditions can be set forth. By application of the asymptotic

expansions for $J(x, y)$, these solutions, including discontinuities arising, are shown to be limiting forms of the general solutions presented in the preceding paper.
N. A. Hall (Minneapolis, Minn.).

Finn, Robert. Isolated singularities of solutions of non-linear partial differential equations. Trans. Amer. Math. Soc. 75, 385–404 (1953).

Cet article est consacré à l'équation aux dérivées partielles

$$(*)\quad \sum_{i=1}^{n} \partial\Theta_i/\partial x_i=0 \quad \text{où} \quad \Theta_i\equiv\Theta_i[x,\varphi(x),\operatorname{grad}\varphi(x)],$$

non linéaire, du second ordre en la fonction inconnue $\varphi(x)$, $x\equiv x_1, \cdots, x_n$. Une solution de (*) est dite admissible dans un domaine D si elle est deux fois continûment différentiable dans D, les Θ_i l'étant alors une fois, et telle que $\sum_{i=1}^{n}\Theta_i(\partial\varphi/\partial x_i)\geqq 0$, l'égalité ayant lieu si grad $\varphi=0$ et seulement dans ce cas. L'auteur établit différents théorèmes relativement généraux quant au comportement d'une solution uniforme de (*), admissible au voisinage d'un de ses points singuliers isolés. Ces théorèmes font intervenir diverses conditions supplémentaires. Ils présentent des conséquences intéressantes: ainsi, $\varphi(x)$ ne peut atteindre de maximum ou de minimum strict dans une région de régularité; citons également une nouvelle démonstration simple d'un résultat dû à L. Bers [Ann. of Math. (2) **53**, 364–386 (1951); ces Rev. **13**, 244] affirmant que toute solution uniforme de l'équation

$$(**)\quad \partial(\rho\partial\varphi/\partial x)/\partial x+\partial(\rho\partial\varphi/\partial y)/\partial y=0,\qquad \rho=[1+(\operatorname{grad}\varphi)^2]^{-1/2}$$

(à laquelle obéit approximativement le potentiel des vitesses d'un fluide compressible qui se meut parallèment à un plan) est bornée dans le voisinage d'un point singulier isolé qui est, en fait, une singularité enlevable.

Le cas de deux dimensions est étudié en détail, les généralisations éventuelles à $n>2$ dimensions sont seulement indiquées dans les grandes lignes. L'auteur donne quelques extensions de ses théorèmes aux solutions multiformes de (*) et applique ses résultats à une équation générale de la dynamique des fluides compressibles animés d'un mouvement plan, (**) avec $\rho=[1-((\gamma-1)/2)(\operatorname{grad}\varphi)^2]^{1/(\gamma-1)}$, γ désignant le rapport des chaleurs spécifiques du fluid
H. G. Garnir (Liège).

***Tihonov, A. N., i Samarskiĭ, A. A. Uravneniya matematičeskogo fiziki. [The equations of mathematical physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 659 pp. 14.80 rubles.

This book consists of three parts: (i) the theory of the equations of mathematical physics; (ii) applications to physical problems; and (iii) special functions. The first part comprises the body of the text. Each of the chapters (except the first) has an appendix which discusses applications to physical problems of the material just presented. The theory of special functions is taken up separately in a special lengthy (about 100 pages) appendix.

The first chapter gives a brief discussion of the classification of second order partial differential equations. Chapters 2, 3, and 4 treat the simplest problems for equations of hyperbolic, parabolic, and elliptic type. The appendices to these chapters take up such topics as the vibrating string and rod, radioactive decomposition, electrostatics, and hydrodynamics.

Chapter 5 is a continuation of chapter 2 and discusses wave propagation in space. The sixth chapter treats heat diffusion in space while chapter 7 continues the discussion of chapter 4 on elliptic equations. Some of the topics considered in the appendices to these chapters are: elasticity, electromagnetic waves, radio waves on the earth's surface, and hollow resonators. Each of the chapters has a set of exercises at the end. The section on special functions develops the theory of Bessel functions, Legendre, Hermite and Laguerre polynomials. A few applications are given.

This book has a wealth of applications, many to topics not usually treated. For example, two applications which the reviewer has not seen elsewhere concern the diffusion of clouds and the influence of radioactivity on the temperature of the earth's core. *M. H. Protter* (Berkeley, Calif.).

Friedrichs, K. O. On the differentiability of the solutions of linear elliptic differential equations. Comm. Pure Appl. Math. 6, 299–326 (1953).

Soit, dans R_n, un opérateur linéaire, aux dérivées partielles, d'ordre $2r$

$$L\equiv \sum_{\Sigma\mu,\,\Sigma\nu=1}^{r} (-1)^{\Sigma\mu}\partial_{\mu_1\cdots\mu_n} a_{\mu_1\cdots\mu_n;\,\nu_1\cdots\nu_n}(x)\partial_{\nu_1\cdots\nu_n};$$

cet opérateur s'applique à un scalaire ou à un vecteur à p dimensions, auquel cas les $a_{\mu_1\cdots\mu_n;\,\nu_1\cdots\nu_n}(x)$ sont des matrices à p^2 éléments. L'auteur qualifie l'opérateur L d'elliptique dans l'ouvert $\mathfrak{R}$ s'il existe deux constantes positives a_0 et a_1 telles que

$$a_0\int_{\mathfrak{R}}\sum_{\Sigma\mu=r}\partial_{\mu_1\cdots\mu_n}\bar{u}\cdot\partial_{\mu_1\cdots\mu_n}u\,d\mathfrak{R}$$

$$\leqq\int_{\mathfrak{R}}\sum_{\Sigma\mu,\,\Sigma\nu=r}\partial_{\mu_1\cdots\mu_n}\bar{u}\cdot a_{\mu_1\cdots\mu_n;\,\nu_1\cdots\nu_n}\partial_{\nu_1\cdots\nu_n}u\,d\mathfrak{R}$$

$$+a_1\int_{\mathfrak{R}}\sum_{\Sigma\mu=1}^{r-1}\partial_{\mu_1\cdots\mu_n}\bar{u}\cdot\partial_{\mu_1\cdots\mu_n}u\,d\mathfrak{R}$$

quel que soit $u \in C_r(\mathfrak{R})$ et identiquement nul hors d'une certaine bande frontière de $\mathfrak{R}$; cette condition généralise le critère habituel d'ellipticité (pour tout $x \in \mathfrak{R}$, on a

$$\sum_{\Sigma\mu,\,\Sigma\nu=r}\partial_{\mu_1\cdots\mu_n}\bar{u}\cdot a_{\mu_1\cdots\mu_n;\,\nu_1\cdots\nu_n}\partial_{\nu_1\cdots\nu_n}u\geqq a_0\sum_{\Sigma\mu=r}\partial_{\mu_1\cdots\mu_n}\bar{u}\cdot\partial_{\mu_1\cdots\mu_n}u$$

et même la condition d'ellipticité forte de M. I. Vishik [Mat. Sbornik N.S. **29**(71), 615–676 (1951); ces Rev. **14**, 174] (i.e. la relation habituelle vérifiée seulement si on remplace les $\partial_{\mu_1\cdots\mu_n}u$ par $\xi_1^{\mu_1}\cdots\xi_n^{\mu_n}u$ quels que soient les scalaires $\xi_1, \cdots, \xi_n$).

L'auteur établit des théorèmes donnant l'ordre de dérivabilité dans un ouvert $\mathfrak{R}$ d'une solution faible de $Lu=f$, L étant elliptique dans $\mathfrak{R}$, en fonction de l'ordre de dérivabilité dans $\mathfrak{R}$ de f et des coefficients de L. Ces propositions précisent des résultats de F. E. Browder [Proc. Nat. Acad. Sci. U. S. A. **38**, 230–235 (1952), p. 232; ces Rev. **14**, 174] et de L. Schwartz [Théorie des distributions, t. I, Hermann, Paris, 1950, pp. 136–138; ces Rev. **12**, 31]. Ces derniers résultats reposent essentiellement sur l'existence et les propriétés d'une solution élémentaire de l'opérateur L. Dans le présent article, l'auteur se propose d'éviter l'intervention des solutions élémentaires. Il démontre ses théorèmes en adaptant à son équation à coefficients variables, une méthode utilisée jusqu'ici pour étudier la dérivabilité des solutions des opérateurs elliptiques à coefficients constants [pour un exposé clair de cette méthode, dans un cas particulier, cf. K. O. Friedrichs, Ann. of Math. (2) **48**, 441–471 (1947); ces Rev. **9**, 255]. Au cours de ses démonstrations,

l'auteur établit des inégalités intéressantes qui estiment, dans un ouvert $\mathcal{R}''\subset\mathcal{R}$, la somme des carrés des modules des dérivées de u d'un ordre donné en fonction d'expressions analogues formées dans $\mathcal{R}'$, $\mathcal{R}''\subset\mathcal{R}'\subset\mathcal{R}$, au moyen de f et des dérivées d'ordre inférieur de u. Les résultats obtenus paraissent cependant exiger un ordre de dérivabilité trop fort pour f, par comparaison aux résultats analogues établis par l'intervention d'une solution élémentaire; en revanche, ils autorisent un ordre de dérivabilité moins élevé pour les coefficients de L.

La dernière partie de l'article esquisse une démonstration de l'existence de la solution d'un problème de Dirichlet généralisé pour l'équation $Lu=f$. L'auteur fait état d'améliorations diverses dues à P. Lax et L. Nirenberg.

H. G. Garnir (Liège).

John, Fritz. Derivatives of continuous weak solutions of linear elliptic equations. Comm. Pure Appl. Math. 6, 327–335 (1953).

Dans un ouvert Ω de R_n, désignons par u une solution faible de $Lu=f$, L étant un opérateur aux dérivées partielles d'ordre m, linéaire, elliptique dans Ω, au sens ordinaire (i.e., la forme associée aux termes d'ordre m est définie pour $x\in\Omega$). L'auteur montre que si u est continue, elle est solution au sens strict dans Ω si $f\in C_{2[n/2]}(\Omega)$ et si les coefficients des dérivées $j$ièmes dans $L\in C_{j+2[n/2]}(\Omega)$. Ce résultat est moins général que les résultats analogues de F. E. Browder et K. O. Friedrichs [cf. l'analyse précédente], mais il est obtenu par des considérations relativement simples et sans faire appel à l'existence d'une solution élémentaire de L. Quand n est impair, la démonstration utilise les propriétés des moyennes sphériques et en particulier la possibilité de représenter u et ses dérivées au moyen de moyennes sphériques pondérées de u. Le cas de n pair se ramène au précédent en utilisant la méthode de descente. L'auteur indique sommairement comment remplacer la continuité postulée pour u par l'intégrabilité locale. *H. G. Garnir*.

Bers, Lipman. Univalent solutions of linear elliptic systems. Comm. Pure Appl. Math. 6, 513–526 (1953).

The author considers a system of linear partial differential equations

$$(1)\qquad \begin{aligned}\phi_x&=A_{11}(x,y)\psi_x+A_{12}(x,y)\psi_y\\ \phi_y&=A_{21}(x,y)\psi_x+A_{22}(x,y)\psi_y\end{aligned}$$

with coefficients A_{ik} defined in an open domain Ω of the $z(=x+iy)$-plane. The system is assumed to be elliptic in Ω, i.e.,

$$4A_{12}A_{21}+(A_{11}-A_{22})^2<0$$

and the coefficients are supposed to have Hölder-continuous partial derivatives of first order in Ω. A function $u(z)$ is called Hölder-continuous in Ω if for any pair of points z_1 and z_2 in a compact subset Σ of Ω the absolute value $|u(z_1)-u(z_2)|\leq K|z_1-z_2|^\alpha$ with constants K and α which may depend on Σ. The following main theorem is derived: There exists a solution of (1) $\omega(z)=\phi(x,y)+i\psi(x,y)$ satisfying the conditions: (a) the Jacobian

$$\begin{vmatrix}\phi_x & \phi_y\\ \psi_x & \psi_y\end{vmatrix}>0$$

in Ω; (b) $\omega(z)$ is a homeomorphism of Ω; (c) at a prescribed point z_0 of Ω one has $\phi=\psi=0$, $\phi_x=1$, $\phi_y=0$. This result is remarkable on account of the fact that no assumptions are made about the domain Ω and the behavior of the coefficients A_{ik} near the boundary of Ω.

An application of the main theorem is made to the study of a quasi-linear equation

$$(2)\qquad A(p,q)r+2B(p,q)s+C(p,q)t=0$$

where

$$p=\phi_x,\quad q=\phi_y,\quad r=\phi_{xx},\quad s=\phi_{xy},\quad t=\phi_{yy}.$$

The coefficients A, B, C are supposed to be defined in a domain Ω of the (p,q)-plane where they possess Hölder continuous partial derivatives of first order and satisfy the ellipticity condition $AC-B^2>0$. It is shown that (2) can be brought into an equivalent form of a conservation law

$$(2')\qquad \frac{\partial\lambda(p,q)}{\partial x}+\frac{\partial\mu(p,q)}{\partial y}=0$$

where $\kappa=\lambda+i\mu$ satisfies conditions corresponding to the conditions (a), (b), (c) of the previous case.

C. Loewner (Stanford, Calif.).

Panyč, O. I. On asymptotic expansion of the solution of a boundary problem. Mat. Sbornik N.S. 32(74), 385–406 (1953). (Russian)

The author considers the asymptotic behaviour for large k of the solution to the following problem: Let T be the volume bounded by a smooth surface S; find the function u such that (1) $\Delta u=0$ outside T, (2) $\Delta u-k^2u=0$ inside T, (3) u together with its normal derivative is continuous across S, (4) u has a singularity of the type $1/4\pi R$ at a certain point P_0 outside T, and (5) u becomes zero at infinity. Put $k=k_0\exp(i\alpha)$ where k_0 is real and $|\alpha|<\pi/4$; then as $k\to\infty$, $\lim u=0$ inside T while, outside T, $\lim u=u_0$ where u_0 is that solution of Laplace's equation which vanishes on S and has the prescribed singularity at P_0. To terms of order $1/k$, u will be approximated outside T by that solution of Laplace's equation having the prescribed singularity at P_0 and which on S satisfies the boundary condition $u_1=\partial u_1/\partial n$. To terms of order $1/k^2$, u will be approximated by u_2 where u_2 is that solution of Laplace's equation having the prescribed singularity at P_0 and satisfying on S the following boundary condition:

$$u_2=\frac{1}{k}\left(1+\frac{\rho}{k}\right)\frac{\partial u_2}{\partial n},$$

where ρ is the mean curvature of S at the point considered.

B. Friedman (New York, N. Y.).

Kupradze, V. D. Boundary problems of the theory of steady elastic vibrations. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 21–74 (1953). (Russian)

The present paper is a survey of results concerning the fundamental boundary-value problems for the vector equation

$$\Delta u+\frac{\lambda+\mu}{\mu}\operatorname{grad}\operatorname{div}u+k_2^2u=0,$$

and is based mainly on the author's book, "Boundary problems of the theory of vibrations and integral equations" [Gostehizdat, Moscow-Leningrad, 1950; these Rev. 15, 318]. The method employed consists in generalizing the ordinary theory of the potential; and, by means of suitable potentials, constructing regular integral equations which are equivalent to the given boundary-value problems. For a discussion of the method reference is made to the earlier review. I. S. Aržanyh [Akad. Nauk Uzbek. SSSR, Trudy Inst. Mat. Meh., vyp. 8 (1951), unavailable for review] has also constructed similar integral equations for the

boundary-value problems of elastic equilibrium. The author points out that these last mentioned integral equations are not regular, but that they are of the singular type which may be handled by using the results of S. G. Mihlin [Uspehi Matem. Nauk (N.S.) 3, 3(25), 29–112 (1948); 8, no. 1(53), 213–217 (1953); these Rev. 10, 305; 14, 762].

J. B. Diaz (College Park, Md.).

Ovsyannikov, L. V. On Tricomi's problem in a class of generalized solutions of the Euler-Darboux equation. Doklady Akad. Nauk SSSR (N.S.) 91, 457–460 (1953). (Russian)

Let S be a Tricomi domain for the equation

$$(*)\qquad yu_{xx}+u_{yy}=0.$$

That is, S consists of an arc K in the half-plane $y\geqq 0$ with endpoints on the x-axis (at A and B) and the two characteristics in $y\leqq 0$ issuing from these endpoints. A generalized solution $u(x, y)$ of (*) is said to be in class P_s if: (1) u is continuous in $\bar{S}$; (2) $u=0$ on K; (3) u is twice continuously differentiable in S_+ (the part of S for which $y>0$) and satisfies (*) there; (4) there exists a function $\nu(x)\in L_2(A, B)$ so that

$$\lim_{h\to 0}\int_{x_h'}^{x_h''} w(x, y)(u_y(x, h)-\nu(x))dx=0$$

for arbitrary w continuous in $\bar{S}_+$; (5) $u(x, y)$ satisfies in S_- the relation

$$u(x, y)=\gamma_1(\eta-\xi)^{2/3}\int_\xi^\eta \frac{u(s, 0)ds}{(\eta-s)^{5/6}(s-\xi)^{5/6}}-\gamma_2\int_\xi^\eta \frac{\nu(s)ds}{(\eta-s)^{1/6}(s-\xi)^{1/6}},$$

where γ_1 and γ_2 are constants.

For functions of this class we may consider $\nu(x)$ as being obtained from u by the operator N. The author considers those curves K for which the operator N can be inverted and for which a representation of $u(x, 0)$ exists in terms of an integral involving the Tricomi kernel. With these restrictions a class of orthonormal eigensolutions is obtained and it is stated that each solution in P_s has the usual representation in a series of eigenfunctions. Using these results the author shows how the Tricomi problem for (*) can be solved for the special case where the boundary values on K vanish.

M. H. Protter (Berkeley, Calif.).

Germain, P., et Bader, R. Solutions élémentaires de certaines équations aux dérivées partielles du type mixte. Bull. Soc. Math. France 81, 145–174 (1953).

Es handelt sich um die "gemischte" partielle Differentialgleichung zweiter Ordnung vom Typus:

$$(*)\qquad \mathfrak{L}(u)=k(z)u_{zz}+u_{xx}=0,$$

deren Koeffizient $k(z)$ eine stetige Funktion in einem Intervall ist, das den Nullpunkt enthält. Diese Funktion $k(z)$ hat das Vorzeichen von z und die Differentialgleichung (*) ist daher elliptisch, wenn z positiv und hyperbolisch, wenn z negativ ist. In der Umgebung des Nullpunktes wird die Entwicklung $k(z)=C_0z+C_1z^2+C_2z^3$ angenommen mit konstanten positiven C_0 und C_1 und einer stetigen Funktion C_2 von z. Im hyperbolischen Bereich sind die Charakteristiken der Differentialgleichung (*) reell und durch $k(z)dz^2+dx^2=0$ definiert. Insbesondere ergibt sich Tricomi's Differentialgleichung aus (*) für $k(z)=z$. Aus (*) entsteht durch Fouriertransformation (nach einer Definition von L. Schwartz) die Differentialgleichung

$$(**)\qquad U_{zz}-4\pi^2\alpha^2k(z)U=0.$$

Mit Hilfe von (**) und zwei ihrer Lösungen $S(\alpha, z)$ und $D(\alpha, z)$, welche den Anfangsbedingungen

$$S(\alpha, 0)=0,\quad D(\alpha, 0)=1,\quad S_z(\alpha, 0)=1,\quad D_z(\alpha, 0)=0$$

genügen, gelingt es den Verfassern festzustellen, ob die zu (*) gehörige Riemannsche Funktion $\mathfrak{R}(M, P)$ quer über den parabolischen Bogen AB (des Kurvendreiecks ABP) in der ganzen elliptischen Halbebene fortgesetzt werden kann (das Kurvendreieck ABP liegt in der hyperbolischen Halbebene, die Bogen PA und PB sind charakteristische Bogen). Insbesondere für $k(z)=1-e^{-2z}$ ergeben sich Lösungen der transformierten Differentialgleichung durch Kombination der Besselschen Funktionen $J_{2\pi\alpha}(2\pi\alpha t)$ und $J_{-2\pi\alpha}(2\pi\alpha t)$. Diese Besselfunktionen, trigonometrische Funktionen und Hankelsche Funktionen gehen in die Integraldarstellung der Riemannschen Funktion des Problems ein. Dabei garantieren die Besselfunktionen die Konvergenz dieser Integrale im Gebiet der elliptischen Halbebene. Doch auch im Dreieck ABP bleiben diese Integrale konvergent. Darüber hinaus kann jetzt auch die Fortsetzung der Riemannschen Funktion in der hyperbolischen Halbebene untersucht werden, deren Verhalten sich dem der Riemannschen Funktion der Differentialgleichung von Tricomi analog erweist. Die Singularitäten der Riemannschen Funktion hängen wesentlich von der Funktion $k(z)$ ab.

In einem zweiten und dritten Abschnitt untersuchen die Verfasser eingehend sogenannte Elementarlösungen der Differentialgleichung (*). Darunter werden Lösungen verstanden, die eine spezielle Singularität, wie sie durch den Typus der Differentialgleichung bestimmt ist, in Evidenz setzen. So im elliptischen Falle etwa eine logarithmische Punktsingularität. Auch hier erweisen sich wieder Fouriertransformationen als vorteilhaft. Der eigentlichen Konstruktion der Elementarlösungen geht eine Untersuchung des asymptotischen Verhaltens der Funktionen $S(\alpha, z)$ und $D(\alpha, z)$ voraus. Als besondere Fälle seien die Funktionen $k(z)=z$, $k(z)=1-e^{-2z}$ und

$$k(z)=\frac{(\gamma+1)\tau-(\gamma-1)}{(\gamma-1)(1-\tau)^{(\gamma+1)/(\gamma-1)}},\quad dz=\frac{(1-\tau)^{1/(\gamma-1)}}{2\tau}d\tau,$$

$$\tau(0)=\frac{\gamma-1}{\gamma+1}$$

erwähnt, deren erste von Tricomi mehrfach behandelt worden ist. Um die Charakterisierung der Singularität durch Elementarlösungen der Gleichung gemischten Typs eindeutig zu machen, sind im allgemeinen zusätzliche Bedingungen erforderlich. Das gilt auch im hyperbolischen Gebiet und wird von den Verfassern unter Aufzählung der verschiedenen möglichen Fälle eingehend diskutiert. Dabei ergeben sich im hyperbolischen Gebiet auch unmittelbare Zusammenhänge mit den aerodynamischen Untersuchungen von M. J. Lighthill [Quart. J. Mech. Appl. Math. 3, 303–325 (1950); diese Rev. 12, 454].

M. Pinl (Dacca).

Agmon, S., Nirenberg, L., and Protter, M. H. A maximum principle for a class of hyperbolic equations and applications to equations of mixed elliptic-hyperbolic type. Comm. Pure Appl. Math. 6, 455–470 (1953).

Cet article étend à une large classe d'équations aux dérivées partielles du type hyperbolique un "principe du maximum" énoncé par Germain et Bader dans le cas très particulier de l'équation de Tricomi [C. R. Acad. Sci. Paris

232, 463–465 (1951); ces Rev. 14, 177]. Un des principaux intérêts de ce principe est de fournir des démonstrations très simples et très générales pour un certain nombre de problèmes aux limites relatifs à une équation du type mixte. Les auteurs commencent par considérer le cas d'une équation linéaire du type hyperbolique rapportée à ses caractéristiques; le domaine auquel s'applique le principe est limité par deux segments de caractéristique concourrants et un arc de courbe coupé en un point au plus par une des familles de caractéristiques. Les coefficients de l'équation doivent satisfaire certaines inégalités; l'une d'elles, qui apparait essentielle, précise le signe de l'un des invariants de Laplace de l'équation. Enfin les fonctions auxquelles s'appliquent le principe doivent être monotones le long d'une des caractéristiques limitant le domaine. On voit donc que de nombreuses conditions sont requises, contrairement à ce qui se passe dans le cas d'une équation de type elliptique. Les auteurs peuvent ensuite se débarrasser de certaines conditions si le principe du maximum cherché s'entend au "sens de A. Haar". Le théorème ainsi obtenu est en un certain sens le meilleur possible, puisqu'il s'énonce sous forme de condition nécessaire et suffisante. Le reste de l'article est consacré aux applications aux équations du type mixte et aux problèmes aux limites correspondants. Les conditions nécessaires à l'application de ce principe restent encore trop sévères pour donner la preuve complète de l'unicité du problème de Tricomi dans le cas de l'équation qui régit les mouvements d'un fluide parfait compressible.

P. Germain (Providence, R. I.).

Kline, Morris. An asymptotic solution of linear second-order hyperbolic differential equations. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. EM-48, i+34 pp. (1952).

The object of this paper is to obtain an asymptotic series for the solution $u(x, t)$, under associated initial and boundary conditions, of the equation

$$(*)\qquad L(u)\equiv \sum_{i,j=1}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{k=1}^{n} b^k\frac{\partial u}{\partial x_k}+cu=0.$$

The symbol x stands for the first $n-1$ variables $x_1, x_2, \cdots, x_{n-1}$, and t for the variable x_n. The coefficients depend on x but not on t. The case of a non-homogeneous term, representing a source or applied force, is also considered; and when this applied force has a time behavior represented by Heaviside's unit function, the resulting solution is $U(x, t)$. The solution due to an applied force with arbitrary time-behavior is then

$$u(x, t)=\frac{\partial}{\partial t}\int_0^t U(x, t-\tau)f(\tau)d\tau,$$

by Duhamel's principle. In particular, if the arbitrary time-behavior is harmonic, viz., $e^{-i\omega t}$, then, neglecting transients, we find that $u(x, t)=\bar{u}(x)e^{-i\omega t}$; and by integrating by parts, we get an asymptotic series for $\bar{u}(x)$ valid for large ω, viz.,

$$\bar{u}(x)=\sum_\alpha[U(x, \tau_\alpha)]e^{i\omega\tau_\alpha}-\frac{1}{i\omega}\sum_\alpha[U_t(x, \tau_\alpha)]e^{i\omega\tau_\alpha}+\cdots.$$

The expression $[U(x, \tau_\alpha)]$ is the saltus of $U(x, t)$ qua function of t at a discontinuity τ_α (which naturally depends on x), and so on.

Although the coefficients in this series are intimately related to the discontinuities of the pulse-solution $U(x, t)$ and of its successive derivatives with respect to t, the primary object of the paper is to show how the coefficients may be obtained without determining $U(x, t)$. For this purpose, the discontinuities of equation (*) are studied. The essential tool is that, if $M(v)$ is adjoint to $L(u)$, the differential equation $L(u)=0$ is equivalent to the integral equation

$$\int_G uM(\Omega)dx_1dx_2\cdots dx_n=0$$

wherein it is understood that u satisfies the integral equation for any Ω which possesses first and second derivatives in G and which vanishes, together with its first derivatives, on the boundary of G. And as the integral equation has a meaning even when u or its derivatives are discontinuous, it is this integral equation which is taken to define u. From it, the transport conditions satisfied by $[u]$, $[u_t]$, $[u_{tt}]$, $\cdots$ are determined.

The work derives from an analogous development carried out by the author for Maxwell's equations [same series of Research Rep. No. EM-24 (1950); these Rev. 12, 886], following ideas introduced by R. K. Luneberg.

E. T. Copson (St. Andrews).

Heaslet, Max. A., and Lomax, Harvard. Further remarks concerning integral transforms of the wave equation. J. Aeronaut. Sci. 21, 142 (1954).

Borowitz, Sidney, and Friedman, Bernard. Three-body scattering problems. Physical Rev. (2) 93, 251–252 (1954).

Evans, George W., II. An application of the Mauro Picone theorem for heat conduction. Proc. Amer. Math. Soc. 4, 961–968 (1953).

Thermocouples, placed in an inside wall ($x=0$) and an outside wall ($x=a$) of a large oven, activate a heating unit located on the outside wall. The temperature $u(x, t)$ between the walls is assumed to satisfy the equation $\alpha u_{xx}=u_t$, and on the wall $x=0$, $u_x(0, t)=A$. Here α and A are constants. If a steady state could be maintained, it would follow that $u=Ax+U$, and the heating unit would maintain $u_x(a, t)=A$. However, the thermocouples can not recognize small changes in temperature, so the control unit operates under the two rules: If $(-1)^ju(0, t)\geqq(-1)^jU+\epsilon$, or if $(-1)^ju(0, t)<(-1)^jU+\epsilon$, but

$$(-1)^j[u(a, t)-u(0, t)]>(-1)^jaA+\gamma,$$

then $u_x(a, t)=A-(-1)^j\sigma$. In the above inequalities ϵ, γ, σ are positive constants, and one set of rules is obtained by letting $j=1$ and the other set by letting $j=2$. Using the maximum principle for parabolic equations [M. Picone, Math. Ann. 101, 701–712 (1929)], the author shows that if initially u satisfies $Ax+U-(\epsilon+\gamma)\leqq u\leqq Ax+U+(\epsilon+\gamma)$, $0\leqq x\leqq a$, then the above control system will hold u in this range of values.

F. G. Dressel (Durham, N. C.).

Fulks, W. The Neumann problem for the heat equation. Pacific J. Math. 3, 567–583 (1953).

If ϑ_3 is the Jacobi theta function, let

$$F(x, t; y, s)=\tfrac{1}{2}\{\vartheta_3(\tfrac{1}{2}(x-y), t-s)+\vartheta_3(\tfrac{1}{2}(x+y), t-s)\}.$$

The author shows that a necessary and sufficient condition that a solution $u=u(x, t)$ of the heat equation $u_{xx}-u_t=0$ be representable in the rectangle R: $(0<x<1, 0<t<1)$ as the sum of $\int_0^1F(x, t; y, 0)dA(y)$, $-\int_0^tF(x, t; 0, s)dB(s)$, and $\int_0^tF(x, t; 1, s)dC(s)$, where $A(s)$, $B(s)$, $C(s)$ are of

bounded variation on $0 \leq s \leq 1$ is that u be continuous on R and satisfy $\lim\sup \int_0^1 |u_x(t,s)|ds < \infty$ as $x \to 0+$, $1-$ and $\lim\sup \int_0^1 |u(y,t)|dy < \infty$, as $t \to 0+$. In which case, A, B, C are uniquely determined up to an additive constant at their points of continuity and their derivatives $A'(x)$, $B'(t)$, $C'(t)$ are the respective limits of $u(x,t)$, $u_x(x,t)$, $u_x(x,t)$ as $t \to 0+$, $x \to 0+$, $x \to 1-$, at points where A', B', C' exist. Another theorem deals with the existence of boundary limits along parabolic arcs. *P. Hartman.*

Visconti, Antoine. Sur une solution des équations du type de diffusion; applications à la théorie de la renormalisation. C. R. Acad. Sci. Paris **236**, 2489–2491 (1953).

The operator equation $U = U_0 + \lambda KU$, where U_0 and K are given operators and λ is a parameter, is transformed into an infinite system of linear equations. It is shown that the solution of this system can be formally represented by the quotient of two power series expansions in λ. For the coefficients of these power series, recursion formulas are given. It is intended to use this method in an investigation of the convergence of renormalization procedures, in particular those used in the treatment of free particle scattering. As an example, the problem of the vacuum in Feynman's theory of the positron [Physical Rev. (2) **76**, 749–759 (1949)], where equations of this type are encountered, is briefly discussed. *E. Gora* (Providence, R. I.).

Mertens, Robert. Sur la résolution en $n^{\text{ième}}$ approximation des problèmes de diffusion multiple. C. R. Acad. Sci. Paris **237**, 1644–1645 (1953).

Integral Equations

Schmeidler, Werner. Algebraische Integralgleichungen. II. Math. Nachr. **10**, 247–255 (1953).

[For part I see Math. Nachr. **8**, 31–40 (1952); these Rev. **14**, 382.] A linear homogeneous integral equation of Fredholm type with a continuous positive kernel has at least one positive eigenvalue with a positive eigenfunction (Theorem of Jentzsch). The main part of the paper deals with the generalization of this theorem to nonlinear integral equations of the type

$$(1) \qquad \mu y^n(s) = L[y] + f(s),$$

where

$$(2) \quad L[y] = \sum_{\alpha_1 + \cdots + \alpha_\nu \leq n} \int_0^1 \cdots \int_0^1 L_{\alpha_1 \cdots \alpha_\nu}(s; t_1, \cdots, t_\nu) \times y^{\alpha_1}(t_1) \cdots y^{\alpha_\nu}(t_\nu) dt_1 \cdots dt_\nu.$$

The author treats first the case (3) $\mu y^n(s) = L_n(y)$ where $L_n(y)$ denotes the homogeneous part of order n ($\alpha_1 + \cdots + \alpha_\nu = n$) of $L[y]$, and proves that (3) has at least one positive eigenvalue μ_0 with corresponding positive eigenfunction if all coefficients of $L_n(y)$ are positive.

In the general case of (1) with all $L_{\alpha_1 \cdots \alpha_\nu}$ positive, it is shown: if $\mu > \mu_0$, then (1) has a positive solution for arbitrary $f(s) > 0$. By well-known properties of linear integral equations this result implies that in the linear case ($n=1$) μ_0 is the greatest positive eigenvalue of (3). It is then proved that for general n equation (3) has no complex eigenvalue μ with $|\mu| = \mu_0$. The remainder of the paper indicates generalizations of the previous results (with a suitable definition of the term eigenvalue) to the case where

$$L(y) = \sum_{\alpha=0}^{n-1} L_\alpha[y] y^\alpha(s)$$

$$L_\alpha[y] = \sum_{\alpha_1 + \cdots + \alpha_\nu \leq n-\alpha} \int_0^1 \cdots \int_0^1 L_{\alpha\alpha_1 \cdots \alpha_\nu}(s; t_1, \cdots, t_\nu) \times y^{\alpha_1}(t_1) \cdots y^{\alpha_\nu}(t_\nu) dt_1 \cdots dt_\nu.$$

As basis for his proofs the author quotes a "generalized theorem of Jentzsch" by D. Morgenstern [Beiträge zur nichtlinearen Funktionalanalysis, Dissertation, Technische Universität Berlin, 1952]. This theorem, which is proved by the use of Schauder's fixed point theorem, asserts the existence of a positive eigenvalue with corresponding positive eigenfunction for a completely continuous (not necessarily linear) operator K which maps positive functions of L^1 into positive functions of L^1 and has the property that the image of some sphere with positive radius is bounded away from 0. [Similar generalizations of the theorem of Jentzsch to theorems of functional analysis, also based on Schauder's theorem, have been given in the linear case by M. A. Rutman [Mat. Sbornik N.S. **8(50)**, 77–96 (1940); these Rev. **2**, 104], and in the nonlinear case by E. H. Rothe [Amer. J. Math. **66**, 245–254 (1944); these Rev. **6**, 71].] *E. H. Rothe* (Ann Arbor, Mich.).

Parodi, Maurice. Analyse symbolique et équations intégrales fonctionnelles. Bull. Sci. Math. (2) **77**, 114–119 (1953).

The author applies operational calculus to the "functional integral equation"

$$(1) \qquad f(\alpha t) + \lambda \int_0^\infty k(x,t) f(x) dx = g(t)$$

and discusses the example

$$(2) \qquad f(\alpha t) + \lambda \int_0^\infty (t/x)^{\nu/2} J_\nu(2(tx)^{\frac{1}{2}}) f(x) dx = g(t).$$

[Reviewer's remark. It is hard to see wherein (1) differs from the well-known special case $\alpha = 1$, or what the author's method does for (2) that a straightforward application of Hankel's inversion formula will not do.] *A. Erdélyi.*

HAVE

***Muskhelishvili, N. I. Singular integral equations. Boundary problems of function theory and their application to mathematical physics.** Translation by J. R. M. Radok. P. Noordhoff N. V., Groningen, 1953. vi+447 pp. 28.50 Dutch florins.

Translation of Singulyarnye integral'nye uravneniya [Gostehizdat, Moscow-Leningrad, 1946; these Rev. **8**, 586; see also these Rev. **11**, 523].

Kim, E. I. On a class of integral equations of the first kind with singular kernel. Doklady Akad. Nauk SSSR (N.S.) **91**, 205–208 (1953). (Russian)

Studied is the equation

$$(1) \qquad \int_0^t \frac{d\tau}{t-\tau} \int_{-\infty}^{\infty} u(\eta,\tau) \sum_1^n A_i(\eta,\tau) \nu d\eta d\tau = f(y,t),$$

$$\nu = \exp\left[-\frac{(y-\eta)^2}{4a_i^2(t-\tau)}\right],$$

where the a_i are constants, the A_i and the derivatives $\partial^2 f(y,t)/\partial y^2$, $\partial f(y,t)/\partial t$ are continuous, bounded. The

major result is that, if $\sum a_i A_i(y, t) \neq 0$, then (1) is reducible to integral equations of the second kind, which can be solved by successive approximations. *W. J. Trjitzinsky.*

Fenyő, István. Sur une classe d'équations intégrales singulières. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **1** (1952), 345–353 (1953). (Hungarian. Russian and French summaries)

The procedure applies to linear integral equations of the first or second kinds, with kernels $k(t, \tau)$, $0 \leqq t, \tau < \infty$, with the property that $\int_0^\infty k(t, \tau) \exp(-pt) dt$ is expressible in the form $F(p) \exp(-\tau \xi(p))$, where $F(p)$ and $\xi(p)$ are independent of τ. Such kernels may be expressed in terms of a series of convolution-powers, and attention was called to them by A. M. Efros [Mat. Sbornik **42**, 699–706 (1935)]. The Laplace-transform of the integral equation then takes a particularly simple form. The procedure is applied to the integral equations

$$\varphi(t) - \lambda \int_0^\infty J_0(2(t\tau)^{\frac{1}{2}}) \varphi(\tau) d\tau = l(t),$$

$$\int_0^\infty \int_{\tau/2\sqrt{t}}^\infty e^{-\xi^2} \varphi(t) d\xi d\tau = l(t),$$

and to a boundary-problem for the one-dimensional diffusion equation. *F. V. Atkinson* (Ibadan).

Satō, Tokui. Sur l'équation intégrale

$$xu(x) = f(x) + \int_0^x K(x, t, u(t)) dt.$$

J. Math. Soc. Japan **5**, 145–153 (1953).

The author studies power-series solutions of the singular Volterra equation (1) $xu(x) = f(x) + \int_0^x K(x, t, u(t)) dt$, where $f(x)$, $K(x, t, u)$ are analytic near $x=0$, $t=0$, $u=c_0$. If $c_0 = f(0) + K(0, 0, c_0)$ and $\lambda = K_u(0, 0, c_0)$ is not a positive integer, then (1) has a unique solution $u(x)$, analytic near $x=0$ and satisfying $u(0) = c_0$. If λ is a positive integer, then (1) has in general no such solution, and if it has one, it has a 1-parameter family of them. Also studied are solutions of the form $u(x) = \sum_{j,k} p_{jk} x^j x^{(\lambda-1)k}$. *M. Golomb.*

Aleksandriiskii, B. I. On the theory of certain linear integro-differential systems. Doklady Akad. Nauk SSSR (N.S.) **91**, 181–184 (1953). (Russian)

The author studies the system

$$\text{(1)} \quad \phi^{(n)}(x) - \sum_0^{n-1} a_i(x) \phi^{(i)}(x) - \lambda \sum_1^m K_i \circ \phi^{(i)} = f(x)$$

$$\left(K_i \circ \phi^{(i)} = \int_a^b K_i(x, s) \phi^{(i)}(s) ds \right),$$

$$\text{(1°)} \qquad \phi^{(j)}(x)|_{x=x_j} = 0 \quad (j = 0, \cdots, n-1),$$

f and the coefficients being possibly complex-valued; if $n \geqq m$, the a_i and f are $L_2(a, b)$ and the $K_i(x, s)$ are $L_2(a \leqq x, s \leqq b)$; when $n < m$, the a_i, K_i, f are $m-n$ times derivable in x, the $a_i^{(m-n)}$, $f^{(m-n)}$ are $L_2(a, b)$ and the $K_i^{(m-n)}(x, s)$ are $L_2(a \leqq x, s \leqq b)$ (here the superscripts denote $\partial^{m-n}/\partial x^{m-n}$). Let $x_0, \cdots, x_{n-1}$ be arbitrary on $[a, b]$; (1) is equivalent to (2) $\phi^{(n)}(x) - Q \circ \phi^{(n)} - \lambda Q' \circ \phi^{(n)} = f(x)$ (subject to (1°)), when $n \geqq m$, and to

$$\text{(3)} \qquad \phi^{(n)}(x) - Q \circ \phi^{(n)} - \lambda \sum_0^{m-n} Q_j' \circ \phi^{(n+j)} = f(x)$$

(subject to (1°)), if $n < m$; here Q, Q', Q_j' are certain functions depending on the x_i, K_i. (1) is equivalent (*) to (4) $u(x) - \lambda Q[u(x)] = R(x)$, where $Q[\cdots]$ is a linear operator, provided that, if ϕ satisfies (1), then $u = \phi^{(n)}$ satisfies (4) and that, if u satisfies (4), then $\phi(x) = \int_{x_0}^x \cdots \int_{x_{n-1}}^x u(x) dx \cdots dx$ satisfies (1). (1) is n.s. (non-singular) if there is a non-characteristic value. There exist s. (singular) systems. A n.s. (1) is equivalent (*) to an equation (4), where the operator $Q[\cdots]$ depends on the kernels Q', Q_j' and R depends on f. The author also finds systems to which a s. (1) is equivalent (*). *W. J. Trjitzinsky* (Urbana, Ill.).

Arešev, M. S. On a linear integro-differential equation. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **9**, 3–14 (1952). (Russian)

It is shown that the substitution

$$z = \sum_{k=0}^{n-1} c_k \frac{(x-c)^k}{k!} + \frac{1}{(n-1)!} \int_c^x (x-s)^{n-1} u(s) ds$$

reduces the integro-differential equation

$$z^{(n)}(x) + \sum_{k=0}^{n-1} a_k(x) z^{(k)}(x) = f(x) + \sum_{k=0}^{m} \int_{\alpha_k}^{\beta_k} K_k(x, t) z^{(k)}(t) dt$$

to a Fredholm equation of the second kind if $m < n$. *M. Golomb* (Lafayette, Ind.).

Ženhèn, O. On the existence of solutions of integro-differential equations. Doklady Akad. Nauk SSSR (N.S.) **91**, 1261–1262 (1953). (Russian)

Under the assumption $|f(x, z)| \leqq c$ $(a \leqq x \leqq b, -\infty < z < \infty)$ and additional conditions which insure that the family $A\varphi = f(x, \lambda \int_a^b K(x, t, \varphi(t)) dt)$, $\varphi \in C$, $|\varphi(t)| \leqq c$ is equicontinuous the author finds that the equation $y = Ay$ and similar integro-differential equations have no more than one solution. *M. Golomb* (Lafayette, Ind.).

Marquet, Simone. Etude mathématique de l'équation de Boltzmann. C. R. Acad. Sci. Paris **237**, 1637–1640 (1953).

The paper deals with a generalization of T. Carleman's [Acta Math. **60**, 91–146 (1933)] and R. Marrot's [J. Math. Pures Appl. (9) **25**, 93–159 (1946); these Rev. **8**, 187] work on the initial-value problem for the Boltzmann integro-differential equation of gas theory. No proofs are given. *L. Van Hove* (Princeton, N. J.).

Functional Analysis, Ergodic Theory

Halperin, Israel. Convex sets in linear topological spaces. Trans. Roy. Soc. Canada. Sect. III. **47**, 1–6 (1953).

This paper contains direct, elementary proofs of the basic separation and support properties of convex sets in linear topological spaces. The results and method are very similar to those of M. H. Stone [Convexity, Chicago, 1946 (mimeographed lecture notes)], N. Bourbaki [Espaces vectoriels topologiques, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953; these Rev. **14**, 880], and the reviewer [Duke Math. J. **18**, 443–466 (1951); these Rev. **13**, 354], but were discovered independently by the author. *V. L. Klee.*

Dieudonné, Jean. Sur les espaces de Montel métrisables. C. R. Acad. Sci. Paris **238**, 194–195 (1954).

A Montel space in Bourbaki's terminology is a locally convex Hausdorff vector space which is barreled ("tonnelé")

and all of whose closed bounded sets are compact. The author answers in the affirmative the question as to whether a metrisable Montel space has a countable base of open sets.
L. Nachbin (Los Angeles, Calif.).

Laugwitz, Detlef. Über Normtopologien in linearen Räumen. Arch. Math. 4, 455–460 (1953).

This paper shows first that if the affine dimension a of a linear space is not finite, then n_e, the cardinal number of non-isomorphic inner-product topologies in L, and n, the cardinal number of non-isomorphic norm topologies in L, are restricted by $c \leq 2^a \leq n_e \leq n \leq 2^c$ if $a \leq c$, the cardinal number of the continuum, and $n_e = n = 2^a$ if $c \leq a$. The rest of the paper notes that the affine and metric dimension numbers of a normed linear space L and of its iterated conjugate spaces and of tensor products of these are all invariant under isomorphisms of L. *M. M. Day.*

Altman, M. On linear functional equations in locally convex linear topological spaces. Studia Math. 13, 194–207 (1953).

This paper extends the Riesz theory of linear functional equations of the Fredholm type to linear convex topological spaces. A part of this has been done by Leray [Acta Sci. Math. Szeged 12, Pars B, 177–186 (1950); these Rev. 12, 32]. $\mathfrak{X}$ being a linear convex topological space, a transformation U on $\mathfrak{X}$ to $\mathfrak{X}$ is completely continuous if there exists a vicinity V of the origin such that $U(V)$ is compact in the sense that every infinite subset has a limiting point. By using the scheme of equivalence classes determined by the pseudo-norm associated with V, it is possible to obtain the theorems of the Riesz theory and base the proofs in this more general case on the Riesz results. In order to obtain the relations between the solutions of a functional equation and its adjoint, the adjoint $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ is defined as the space of sequentially continuous linear functionals X on $\mathfrak{X}$, and the adjoint $\bar{U}$ by the usual identity $[\bar{U}(X), x] = [X, U(x)]$ for all x. Then $\bar{U}$ is said to be completely continuous if it maps a bounded set into a compact one, where a set is bounded if there exists a vicinity V of the origin in $\mathfrak{X}$, and a constant M such that on V: $|X(x)| < M$ for all X of the set, and a set is compact if every infinite subset contains a sequence which converges strongly to some X of $\bar{\mathfrak{X}}$ in the sense that there exists a V in $\mathfrak{X}$ such that $\lim_n |X_n(x) - X(x)| = 0$ uniformly on V. In other words, the topology on $\bar{\mathfrak{X}}$ is a sort of uniform weak topology. Then it is shown that if U is completely continuous on $\mathfrak{X}$ to $\mathfrak{X}$ then $\bar{U}$ is completely continuous on $\bar{\mathfrak{X}}$ to $\bar{\mathfrak{X}}$. The usual theories relating the solvability of $x - U(x) = y$ and $X - \bar{U}(X) = Y$ are then deduced largely from the corresponding theorems valid in normed linear spaces by the pseudo-norm method indicated above.
T. H. Hildebrandt (Ann Arbor, Mich.).

Altman, M. Mean ergodic theorem in locally convex linear topological spaces. Studia Math. 13, 190–193 (1953).

This note proves that if T is a linear weakly completely continuous transformation on a locally convex topological space $\mathfrak{X}$ into itself, such that $T^n(x)$ is bounded for each x, then $(\sum_{i=0}^{n-1} T^i(x))/n$ converges for each x of X to a point y such that $y = T(y)$. This is an extension to convex linear topological spaces of the Yosida extension [Proc. Imp. Acad. Tokyo 14, 292–294 (1938)] of v. Neumann's ergodic theorem. Weak convergence is, of course, in terms of the adjoint space $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ of sequentially continuous linear functionals on $\mathfrak{X}$. Weak complete continuity requires the existence of a vicinity V of the origin such that $T(V)$ is sequentially compact in the sense of weak convergence. The use of the pseudo-norm determined by V makes it possible to reduce the proof of the theorem to the Yosida theorem. Basic is the lemma that if T is weakly completely continuous on $\mathfrak{X}$, then corresponding to any pseudo-norm $|x|_\theta$ there exists an M_θ such that $|T(x)|_\theta \leq M_\theta |x|$, where $|x|$ is the pseudo-norm determined by V. *T. H. Hildebrandt.*

Krotkov, Valentina, and Halperin, Israel. The ergodic theorem for Banach spaces with convex-compactness. Trans. Roy. Soc. Canada. Sect. III. 47, 17–20 (1953).

A Banach space is called "convex-compact" provided every decreasing sequence of non-empty bounded closed convex subsets of it has a non-empty intersection. Theorem: For a bounded linear operator T on a convex-compact B, the following two assertions are equivalent: (a) T is ergodic, i.e., the sequence $\{T_n x\}_1^\infty$ is norm-convergent for each $x \in B$ where $T_n = (n+1)^{-1} \sum_{i=0}^n T^i$; (b) $\{\|T_n\|\}_1^\infty$ is bounded and $\{n^{-1} T^n x\}_1^\infty$ converges to 0 for each $x \in B$. Since (as the author remarks) every reflexive space is convex-compact, this includes the ergodic theorem of E. R. Lorch [Bull. Amer. Math. Soc. 45, 945–947 (1939); these Rev. 1, 242] and a theorem of Yosida and Kakutani [Proc. Imp. Acad. Tokyo 14, 333–339 (1938)]. Also discussed is the ergodicity of a certain type of linear operator in spaces L^1. [Reviewer's note: By results of Šmulian [Mat. Sbornik N.S. 5(47), 317–328 (1939); these Rev. 1, 335] and Eberlein [Proc. Nat. Acad. Sci. U. S. A. 33, 51–53 (1947); these Rev. 9, 42; 10, 855], convex-compactness is actually equivalent to reflexivity, so the author's result is essentially included in those mentioned above, and in the more general theorem of Eberlein [ibid. 34, 43–47 (1948); these Rev. 9, 359].
V. L. Klee (Seattle, Wash.).

Bartle, Robert G. Singular points of functional equations. Trans. Amer. Math. Soc. 75, 366–384 (1953).

Let x, y denote points of the Banach-spaces X, Y resp. Let L be a linear map $X \to X$, and $F(x, y)$ a continuous map of a neighborhood of the zero element of the product space $X \times Y$ into X with $F(0, 0) = 0$. The object of the paper is the discussion of solutions of the equation

(1) $$L(x) + F(x, y) = 0$$

neighboring the solution $x = y = 0$ under the assumption (H1): the range of L is closed, and the nullspaces $\mathfrak{U}$ and $\mathfrak{U}^*$ of L and its adjoint L^* are of finite dimension n. ((H1) is satisfied, e.g., if $I - L$, or some power of it, is completely continuous with I denoting the identity.) A further assumption, (H2), concerns continuity and "second order" conditions on $F(x, y)$. Under assumptions (H1), (H2), (1) is transformed into the system

(2) $$x + RF(x, y) = u,$$

(3) $$f_i(F(x, y)) = 0 \quad (i = 1, 2, \cdots, n)$$

where u is in $\mathfrak{U}$, and $f_1(x), \cdots, f_n(x)$ form a base for $\mathfrak{U}^*$, while R is defined as follows: let $u_1, \cdots, u_n$ be a base of U, and let the elements $z_1, \cdots, z_n$ of X and the functionals $g_1(x), \cdots, g_n(x)$ be defined in such a way that $g_i(u_j) = f_i(z_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol. R is then the everywhere defined inverse of $L(x) - \sum_{i=1}^n g_i(x) z_i$. For (small enough) fixed y and u, (2) has a unique solution $x = V(u, y)$ as is seen from the Graves-Hildebrandt implicit function theorem. Since u is an element of an n-dimensional vector space, a system of n equations for the n unknown

components of u as functions of y, the so-called "Verzweigungsgleichung", is obtained if the solution $x=V(u, y)$ of (2) is substituted in (3).

As the author remarks, his paper is closely connected with an earlier one by J. Cronin [same Trans. 69, 208–231 (1950); these Rev. 12, 716] where (1) is treated (with completely continuous $I-L$). Thus the decomposition of (1) into (2) and (3) is essentially the one used by J. Cronin. Indeed, if we set $x=x'+u$, (2) becomes identical with $x'+RF(x'+u, y)=0$ which is equation (3.3) of J. Cronin's paper. However, the proofs for the decomposition given by the two authors are different.

The remainder of the paper deals mainly with a discussion of the "Verzweigungsgleichung" in the case $n=1$, and with applications of the theory to nonlinear integral-equations and ordinary differential equation problems reducible to such equations. *E. H. Rothe* (Ann Arbor, Mich.).

Krasnosel's'kiĭ, M. O. Vector fields symmetric with respect to a subspace. Dopovidi Akad. Nauk Ukrain. RSR **1951**, 8–11 (1951). (Ukrainian. Russian summary)

Let the Banach space E be the Cartesian product of the Banach spaces E_1, E_2. The point $z^*=(x, -y)\in E$, $x\in E_1$, $y\in E_2$ is said to be symmetric to $z=(x, y)$ relative to E_1. The main result of the paper is the following theorem. Let Φ be a completely continuous vector field on some sphere $S\subset E$ such that neither Φ nor its projection $P_1\Phi$ onto E_1 vanishes anywhere. Assume that in no point $z\in S$, Φz^* is parallel to the vector that is symmetric to Φz relative to E_2. Then the index of Φ on S differs from that of $P_1\Phi$ on $S\cap E_1$ by an even integer. This theorem is used to establish the existence of a solution for a certain pair of nonlinear integral equations in two unknown functions. The proofs are said to be based on the results of a paper by the author and M. G. Kreĭn [Ukrain. Mat. Žurnal 1, no. 2, 99–102 (1949); these Rev. 14, 72] that was not accessible to the reviewer. *M. Golomb* (Lafayette, Ind.).

Kato, Tosio. Integration of the equation of evolution in a Banach space. J. Math. Soc. Japan 5, 208–234 (1953).

The author has obtained a solution to the differential equation $dx(t)/dt=A(t)x(t)+f(t)$ and the associated homogeneous equation under very general assumptions by a product integration method. Here the unknown is a function defined for $t\in[a, b]$ with values in a complex Banach space X, $f(t)$ is a given vector-valued function, and for each $t\in[a, b]$, $A(t)$ is a closed linear operator with dense domain whose resolvent $R(\lambda; A(t))$ exists and satisfies the condition $\|\lambda R(\lambda; A(t))\|\leqq 1$ for all $\lambda>0$; that is, $A(t)$ is the infinitesimal generator of a strongly continuous semi-group of contraction operators. This suffices to prove that the differential equation has at most one solution. In order to establish the existence of a solution further assumptions are required. If the domain D of $A(t)$ is independent of t, if $B(t, s)=[I-A(t)][I-A(s)]^{-1}$ is bounded in norm for $a\leqq s, t\leqq b$, and if $B(t, s)$ is strongly continuously differentiable in t for some s, then the homogeneous equation has a strongly continuously differentiable solution for each initial value $x\in D$. The solution at any time $t\in[a, b]$ is a linear function of the initial value of norm $\leqq 1$. In addition, if $[I-A(r)]f(t)$ is strongly continuous in t for some $r\in[a, b]$, then the inhomogeneous equation has a strongly continuously differentiable solution for each initial value $x\in D$. If X is uniformly convex and $B(t, s)$ is merely of uniform bounded variation and continuous in t for some s, then there exists a function whose right-hand derivative satisfies the homogeneous differential equation for initial value $x\in D$. In particular, if X is a Hilbert space and $iA(t)$ is self-adjoint, then this weaker assumption actually implies the existence of a solution in the ordinary sense. This problem has also been treated by the reviewer [Trans. Amer. Math. Soc. 74, 199–221 (1953); these Rev. 14, 882] whose work is not limited to infinitesimal generators of contraction semi-groups, but is otherwise less general than the above. *R. S. Phillips* (New Haven, Conn.).

Kračkovskiĭ, S. N. On properties of a linear operator connected with its generalized Fredholm region. Doklady Akad. Nauk SSSR (N.S.) 91, 1011–1013 (1953). (Russian)

In a previous paper [same Doklady (N.S.) 88, 201–204 (1953); these Rev. 14, 1095] the author has associated with each eigenvalue of a bounded linear transformation A of a complex Banach space into itself integers s, p and r_k. In this paper he defines analogous integers t, q and ν_k for the adjoint operator A^* and shows that $p=q$ and $r_k=\nu_k$, $k=1, \cdots, p$, and that the numbers s and t are constant for λ's from the same component of the generalized Fredholm region S_A (i.e. the set of λ's for which $I-\lambda A$ is a generalized Fredholm operator [see the review of Atkinson, Mat. Sbornik (N.S.) 28(70), 3–14 (1951); these Rev. 13, 46]). It was known that the index of T_λ $(=s-t)$ is constant in each component of S_A. *J. V. Wehausen* (Providence, R. I.).

Erdös, P., and Straus, E. G. On linear independence of sequences in a Banach space. Pacific J. Math. 3, 689–694 (1953).

The following problem of Dvoretzky is solved. Let $\{x_n\}$ be unit vectors in a Banach space and assume that they are linearly independent ($\sum_1^n \alpha_m x_m=0$ implies $\alpha_m=0$). Does there exist a subsequence $\{x_{n_i}\}$ which is linearly independent in one of three senses: (I) $\sum_1^\infty \alpha_{n_i}x_{n_i}=0$ implies $\alpha_{n_i}=0$; (II) for any function $\varphi(n)>0$, $|\alpha_{n_i}^{(k)}|<\varphi(n_i)$ and

$$\lim_{k\to\infty}\sum_1^\infty \alpha_{n_i}^{(k)}x_{n_i}=0 \text{ implies } \lim_{k\to\infty}\alpha_{n_i}^{(k)}=0;$$

$$\text{(III)}\qquad \lim_{k\to\infty}\sum_1^\infty \alpha_{n_i}^{(k)}x_{n_i}=0 \text{ implies } \lim_{k\to\infty}\alpha_{n_i}^{(k)}=0?$$

The set $(x+\lambda_n)^{-1}$ on $[0, 1]$, $\lambda_n>0$, $\lambda_n\to\infty$ is a counter-example for (III). The main theorem is: Let $\{x_n\}$ be linearly independent, $|x_n|=1$, $\varphi(n)>0$. Then there is a subsequence $\{x_{n_i}\}$ such that $|\alpha_{n_i}^{(k)}|<\varphi(n_i)$ and $\lim_{k\to\infty}\sum_1^\infty \alpha_{n_i}^{(k)}x_{n_i}=0$ implies $\lim_{k\to\infty}\alpha_{n_i}^{(k)}=0$. Use is made of the embeddability of $[x_n]$ in $C(0, 1)$ and orthonormalization of $\{x_n\}$ (considered in $L_2(0, 1)\supset C(0, 1)$, together with neat estimates on coefficient sizes. Misprints: (1) p. 689 read "$|c_n^{(k)}|<\varphi(n)$" for "$|c_n^{(k)}|<\varphi(k)$"; (2) p. 690, l. 11 from bottom, read "x_n" for "x_{n_i}"; (3) p. 691, l. 5 from bottom, read "$\lim_{n\to\infty}\sum_{k=1}^\infty c_k^{(n)}y_k=0$" for "$\lim_{n\to\infty}\sum_{k=1}^\infty c_k^{(n)}y_k=Q$". *B. R. Gelbaum* (Minneapolis, Minn.).

Šilov, G. E. Criteria of compactness in a homogeneous space of functions. Doklady Akad. Nauk SSSR (N.S.) 92, 11–12 (1953). (Russian)

Let G be a compact Abelien group, written additively, and let R be a normed complex linear space of complex-valued functions on G. For each $f\in R$ and $h\in G$, let f_h be the translated function $f(t+h)$, $t\in G$. Suppose R satisfies the following two conditions. (a) If $f\in R$ and $h\in G$, then $f_h\in R$ and $\|f_h\|=\|f\|$. (b) For each $f\in R$, $\lim_{h\to 0}\|f_h-f\|=0$.

(Such spaces have previously been considered by the author [Uspehi Matem. Nauk. (N.S.) 6, no. 1(41), 91–137 (1951); these Rev. 13, 139].) Theorem: For precompactness (=total boundedness) of a set $M \subset R$ it is necessary and sufficient that M is bounded and the convergence in (b) above is uniform over M. Proof of sufficiency uses the author's version [loc. cit., p. 99] of the Bochner-von Neumann generalization of Fejér's theorem on summability of trigonometric series. (For the case $R = L^p$ $(1 \leq p \leq \infty)$, the author's theorem follows from a compactness criterion of Weil, valid for arbitrary locally compact G [L'intégration dans les groupes topologiques et ses applications, Hermann, Paris, 1940, p. 53; these Rev. 3, 198].) *V. L. Klee.*

Grothendieck, Alexandre. Sur certains espaces de fonctions holomorphes. I. J. Reine Angew. Math. 192, 35–64 (1953).

The author says that this paper is intended to generalize and complement Köthe's study of duality in function-theory [same J. 191, 30–49 (1953); these Rev. 15, 132], but that the paper was conceived independently of the work of Köthe, after reading one of the papers of J. Sebastião e Silva [Portugaliae Math. 9, 1–130 (1950); these Rev. 11, 524].

The paper is in two installments, of which this is the first. It deals throughout with locally convex linear topological spaces, after the manner of Bourbaki and the modern French school. A space of this kind is hereafter abbreviated as an LCS.

First there is a brief resumé of the theory of analytic functions of a complex variable with values in some LCS (with complex scalars). No originality is claimed by the author for this theory, but the reviewer found a number of new insights in the exposition, which is certainly not well known enough to be considered standard. The situation where the LCS is a Banach space has been well known for a long time, of course. Next there is an examination of the meaning of analyticity when the LCS is any one of six different spaces. In most of these cases the LCS is a space of functions, and a function of z with values in the space is envisaged as a function $f(z, t)$ such that when z is fixed and t is the variable, $f(z, t)$ is a member of the space. The situation is simplest when the function space E is the space of continuous functions on a locally compact space (with the topology of compact convergence for E). Analyticity in this case means continuity jointly in z and t and analyticity in z for each fixed t. Consideration of functions of this kind (for M a compact interval of the real axis) goes back to Bôcher [Ann. of Math. (2) 12, 18–26 (1910)], and their significance in the theory of vector analytic functions was recognized by the reviewer [Bull. Amer. Math. Soc. 49, 652–669 (1943), esp. p. 655; these Rev. 5, 39; Ann. of Math. (2) 39, 574–593 (1938), esp. p. 579]. If E is an L^p space over a locally compact space with a positive measure, the condition of analyticity is that f be continuous as a function with values in E and that there exist a function $g(z, t)$ analytic in z for each t such that, for each z, $f(z, t) = g(z, t)$ locally almost everywhere. The reviewer wishes to point out that he gave a theorem of similar nature characterizing analytic functions with values in L^p (for ordinary Lebesgue measure) in 1937 [ibid., p. 581]. The case L^2 had been partially examined even earlier by Wiener [Fund. Math. 4, 136–143 (1923)].

The author proceeds next to consider spaces of analytic functions and to investigate the representations of linear functionals and linear transformations defined on such spaces. The subject matter here is similar to that of Silva and Köthe, but the treatment and the results are more general. If Ω_1 is any non-void subset of the Riemann sphere Ω, with $\Omega_1 \neq \Omega$, and if $P(\Omega_1, E)$ is the class of all "locally analytic" functions on Ω_1 with values in an LCS E, $P(\Omega_1, E)$ becomes an LCS with a suitably defined topology (which, if Ω_1 is open, is the topology of compact convergence). If $\Omega_2 = \Omega - \Omega_1$ and E' is the dual of E, the dual of $P(\Omega_1, E)$ is the subspace of $P(\Omega_2, E')$ consisting of those functions which have locally an equicontinuous image in E'. In particular, if E is metrizable and complete, or if it is a t-space (French: espace tonnelé), the dual of $P(\Omega_1, E)$ is all of $P(\Omega_2, E')$.

A bounded linear mapping of one LCS into another is continuous, but continuity does not always imply boundedness. Let E and F each be an LCS, let F be complete, and let E be such that the closed, convex, circled (French: cerclée) envelope of its compact sets are compact. Let $\mathfrak{L}_s(E, F)$ be the space of linear continuous mappings of E into F, with the topology of simple convergence. Let Ω_1 and Ω_2 be nonempty complementary subsets of Ω. Then there is a 1-1 correspondence between the class of bounded linear mappings of $P(\Omega_1, E)$ into F and a certain subclass of $P(\Omega_2, \mathfrak{L}_s(E, F))$, which is all of the latter space if E is metrizable and complete and F is a Banach space. The foregoing representation theorem can be modified so as to yield all the continuous (not merely the bounded) linear mappings of $P(\Omega_1, E)$ into F if Ω_1 is compact and E and F both satisfy the condition previously imposed on E.

Finally, by generalizing the definition of a locally analytic function on a set in Ω and replacing the space $P(\Omega_2, F)$ by a space $G(\Omega_2, F)$ of functions locally analytic on Ω_2 in the "wide sense," it is shown that the class of continuous linear mappings of $P(\Omega_1)$ into F is in 1-1 correspondence with $Q(\Omega_2, F)$. Here $P(\Omega_1)$ is $P(\Omega_1, E)$ with E the space of complex numbers, and F is any complete LCS. The first installment concludes with a discussion of the significance of analyticity in the wide sense for various particular realizations of F. *A. E. Taylor* (Los Angeles, Calif.).

Grothendieck, A. Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. Canadian J. Math. 5, 129–173 (1953).

This paper is a step in the direction of organizing into a general theory known results concerning continuous linear mappings between special Banach spaces, such as $C(K)$ (space of all complex-valued continuous functions on a compact K), $\mathfrak{M}^1(K)$ (dual of $C(K)$, space of all Radon measures on K), $L^1(\mu)$ (space of all complex-valued functions integrable with respect to a Radon measure μ on K). A product of two continuous linear mappings is usually of a more special type than the factors themselves. Accordingly the author gives some cases in which this vague statement actually holds. The author's considerations are motivated by an article by Dunford and Pettis [Trans. Amer. Math. Soc. 47, 323–392 (1940); these Rev. 1, 338]. The locally convex (complex vector) space E is said to have the Dunford-Pettis (D.-P.) property if every continuous linear mapping of E into any locally convex Hausdorff space F which maps bounded sets onto weakly relatively compact sets necessarily maps circled convex weakly compact sets onto strongly compact sets. E has the strict D.-P. property if every continuous linear mapping of E into any F which maps bounded sets onto weakly relatively compact sets necessarily maps weak Cauchy sequences onto strong

Cauchy sequences. Alternative ways of defining these properties which involve only E and its duals E', E'' are given. In case the strong dual E' has the D.-P. property, the same is true for E if strongly bounded sets in E' are equicontinuous (which is the case if E is a Fréchet space). For a Fréchet E, the strict D.-P. property is stronger than the D.-P. property and actually equivalent to it if every weak Cauchy sequence is weakly convergent. If E has the D.-P. property and u, v are endomorphisms of E mapping bounded sets onto weakly relatively compact sets, then uv maps bounded sets onto strongly relatively compact sets. If E, F, G are locally convex spaces, E and F complete and one of them with the D.-P. property, $u: E\times F\to G$ bilinear continuous, $x_i\to x_0$ weakly in E, $y_i\to y_0$ weakly in F, then $u(x_i, y_i)\to u(x_0, y_0)$ weakly in G. Every $C(K)$ and $L^1(\mu)$ has the strict D.-P. property; same for the space $\mathfrak{E}^{(m)}(O)$ of all m-continuously differentiable complex-valued functions in an open set $O\subset R^n$. E is said to have the reciprocal D.-P. property if every continuous linear mapping of E into any complete locally convex Hausdorff space which maps circled convex weakly compact sets onto strongly compact sets necessarily maps bounded sets onto weakly relatively compact sets. The reciprocal strict D.-P. property is defined similarly. The spaces $C(K)$ and $\mathfrak{E}^{(m)}(O)$ are shown to have the reciprocal D.-P. property. $L^1(\mu)$ fails to have it in general. The author gives several necessary and sufficient conditions for a subset of $\mathfrak{M}^1(K)$ to be weakly relatively compact [cf. J. Dieudonné, Anais Acad. Brasil. Ci. **23**, 21–38 (1951); these Rev. **13**, 121] and remarks that each such set of conditions gives by transposition a criterion for a continuous linear mapping on $C(K)$ to be weakly completely continuous (weakly compact in the author's terminology). Next E is said to have the Dieudonné (D.) property if every continuous linear mapping of E into any complete locally convex space which maps weak Cauchy sequences onto weakly convergent sequences necessarily maps bounded sets onto weakly relatively compact sets. Every semi-reflexive space has property D. If E has property D and F is locally convex complete such that every weak Cauchy sequence is weakly convergent, then obviously every continuous linear mapping $E\to F$ maps bounded sets onto weakly relatively compact sets. $C(K)$ has property D. Every continuous linear mapping of $C(K)$ into some $L^1(\mu)$ is weakly completely continuous. Let E, F be locally convex spaces. A bilinear functional u on $E\times F$ is called integral in case there exist weakly closed equicontinuous sets $A'\subset E'$, $B'\subset F'$, hence both weakly compact, and a Radon measure μ on the compact space $A'\times B'$ such that

$$u(x, y)=\int_{A'\times B'}\langle x, x'\rangle\cdot\langle y, y'\rangle d\mu(x', y').$$

A linear continuous mapping $E\to F$ is said to be integral if the associated bilinear functional on $E\times F'$ (F' strong) is integral. For complete spaces, every integral mapping is weakly completely continuous and the product of two integral mappings is completely continuous. For Banach spaces E, F, an integral mapping is completely continuous if one of the two spaces is reflexive. As further applications of the methods of this paper, the author proves that, for a compact K and a locally convex complete E, the (weakly or strongly) completely additive E-valued functions of Borelian sets correspond in a one-to-one way to the weakly completely continuous linear mappings of $C(K)$ into E and proves some special results for K Stonian and for $K=c_0$.

L. Nachbin (Los Angeles, Calif.).

Grothendieck, A. Sur certains sous-espaces vectoriels de L^p. Canadian J. Math. **6**, 158–160 (1954).

The author remarks that, if M is a space with a bounded measure μ and $1\leqq p<+\infty$, then every vector subspace of $L^\infty(\mu)$ closed in $L^p(\mu)$ is necessarily finite-dimensional. A few other results along the same line are also mentioned.

L. Nachbin (Los Angeles, Calif.).

Vaĭnberg, M. M. On the structure of an operator. Doklady Akad. Nauk SSSR (N.S.) **92**, 213–216 (1953). (Russian)

Let B be a measurable set in Euclidean n-space, and let L^p, $p>0$, be the space of measurable real-valued functions u on B for which $\|u\|=\int_B|u(x)|^p dx$ is finite. Let $f(u, x)$ be a real-valued function defined for u real and x in B so that it is continuous for each fixed x and measurable for each fixed u. Define h by $hu(x)=f(u(x), x)$. Theorem. In order that h carry L^p into L^{p_1}, where $p>0$ and $p_1>0$, it is necessary and sufficient that there exist φ in L^{p_1} and $b>0$ such that for all real u and x in B, $f(u, x)\leqq\varphi(x)+b\|u\|^r$, where $r=p/p_1$.

M. M. Day (Urbana, Ill.).

Ellis, H. W., and Halperin, Israel. Function spaces determined by a levelling length function. Canadian J. Math. **5**, 576–592 (1953).

A length function $\lambda(u)$ is a non-negative function, subadditive, positive homogeneous, monotone increasing and satisfying certain continuity and σ-finiteness conditions, which is defined over the set $S(E, \gamma)$ of real-valued non-negative functions defined over E and measureable with respect to a completely additive measure γ. The conjugate length function λ is defined by $\lambda^*(u)=\sup\int_E u(p)v(p)d\gamma$ where the supremum is taken with respect to all v with $\lambda(v)\leqq 1$. The linear space L^λ ($L^\lambda(B)$) of functions real (vector) valued for which $\lambda(|u|)<\infty$ is shown to be a Banach space with norm defined by $\lambda(|u|)$. λ is a levelling length function if (a) there exist sets e, e' with $\lambda(\psi_e)>0$, $\lambda(e')<\infty$ where ψ_e is the characteristic function of e and (b) if $u(P)=\sum_{i=1}^n k_i\psi_{e_i}(P)$ is any simple function, $k_i\geqq 0$ ($i=1, \cdots, n$), then

$$\lambda(u)\geqq\lambda\left[\frac{k_1\gamma(e_1)+k_2\gamma(e_2)}{\gamma(e_1+e_2)}\psi_{e_1+e_2}+\sum_{i=2}^n k_i\psi_{ei}\right].$$

For a levelling function L^λ, $L^\lambda(B)$ are generalizations of L^p, $L^p(B)$ and λ^* corresponds to the conjugate index to p, $1\leqq p\leqq\infty$. The authors prove for L^λ, $L^\lambda(B)$ theorems similar to known theorems for L^p, $L^p(B)$, for example, a Hölder-type inequality, an equivalence between spaces L^λ and spaces V^λ of set functions defined similarly to the Riesz spaces V^p, and Riesz type representation theorems for linear functionals over L^λ, $L^\lambda(B)$ in terms of integrals and functions in L^{λ^*}, $L^{\lambda^*}(B)$. In the case of vector-valued functions, the relationship of $L^\lambda(B)$ and $L^{\lambda^*}(B^*)$ is investigated and conditions which imply their equality are given thus generalizing similar theorems which are known for L^p, $L^p(B)$. The spaces $L^p_{(w)}(B)$, $M^q_{(w)}(B)$ of Halperin [same J. **5**, 273–288 (1953); these Rev. **15**, 38] are investigated as special cases.

R. E. Fullerton (Madison, Wis.).

Krasnosel'skiĭ, M. A. Application of variational methods to the problem of branch points. Mat. Sbornik N.S. **33**(75), 199–214 (1953). (Russian)

Suppose A is a nonlinear operator in a real Hilbert space H, which leaves the zero element θ invariant. If $A\varphi_0=\lambda_0\varphi_0$, $\varphi_0\neq\theta$, then λ_0 is an eigenvalue, φ_0 an eigenvector of A.

λ_0 is a branch point of A if for every positive ϵ, δ there exists an eigenvalue λ and eigenvector φ for which $|\lambda-\lambda_0|<\epsilon$, $\|\varphi\|<\delta$. The main result of this paper is contained in the following theorem. Suppose the completely continuous operator Γ ($\Gamma\theta=\theta$) is the gradient of a weakly continuous functional Φ ($\Phi(\theta)=0$), and Γ has at θ a Fréchet derivative B (linear) which is completely continuous and self-adjoint. Then every eigenvalue of B is a branch point of Γ. The proof of this theorem is based on a minimax construction for the functional Φ, along the lines developed by Lyusternik and Šnirelman [Uspehi Matem. Nauk (N.S.) **2**, no. 1(17), 166–217 (1947); these Rev. **10**, 624]. The results are applied to the nonlinear integral equation $\lambda\varphi(x)=\int_G K(x,y)f[y,\varphi(y)]dy$, $f(y,0)\equiv 0$. *M. Golomb* (Lafayette, Ind.).

Krasnosel'skiĭ, M. A., and Povolockiĭ, A. I. On variational methods in the problem of branch points. Doklady Akad. Nauk SSSR (N.S.) **91**, 19–22 (1953). (Russian)

The authors announce several results concerning branch points of nonlinear operators (for terminology see the preceding review), of which the following two are typical. (I) Suppose Γ is as in the paper reviewed above with the additional hypothesis that B is positive-definite. Also suppose J is a unitary operator that commutes with B, and JB has a finite number of positive eigenvalues. Then the smallest positive eigenvalue of B is a branch point of J. (II) Suppose H is the orthogonal sum of H_1 and H_2, where H_1 is finite-dimensional, and J is defined by $J(x+y)=x-y$ ($x\in H_1$, $y\in H_2$). Suppose the weakly continuous functional is such that in the subset $\|x\|\geqq\|y\|$, $\lim_{\|x+y\|\to\infty}\Phi(x+y)=+\infty$, and the completely continuous operator Γ ($\Gamma\theta=\theta$) is the gradient of Φ. Then $J\Gamma$ has at least a denumerably infinite number of eigenvectors, among them such of arbitrarily large norm. It is hinted that the proofs are based on the minimization (or maximization) of Φ on the hyperboloid $(J\varphi,\varphi)=\text{const}$. *M. Golomb.*

Kračkovskiĭ, S. N., and Gol'dman, M. A. Some properties of a completely continuous operator in Hilbert space. Latvijas PSR Zinātņu Akad. Vēstis **1950**, no. 10(39), 93–106 (1950). (Russian. Latvian summary)

For the most part this paper provides proofs for theorems announced elsewhere [Doklady Akad. Nauk SSSR (N.S.) **70**, 945–948 (1950); these Rev. **11**, 600 (we follow the notation of this review)]. Additional material includes a discussion of the "absolute norm" $N(\mathfrak{A})$ of a completely continuous operator. Here $N^2(\mathfrak{A})=\sum_1^\infty\|\mathfrak{A}x_p\|^2$ where x_p is any closed orthonormal set of elements of H [cf. Smirnov, A course of higher mathematics, vol. 5, Gostehizdat, Moscow-Leningrad, 1947, p. 392 ff.; these Rev. **9**, 574]. It is shown that $N(\mathfrak{A}_2)<N(\mathfrak{A})$ if $\mathfrak{A}_2\neq 0$ and $N(\mathfrak{A}_2)<\infty$, that $N^2(\mathfrak{A}_1)\geqq\sum_1^\infty|\lambda_p|^{-2}$, where the λ_p's are eigenvalues of $\mathfrak{A}$ and occur with a multiplicity equal to the dimension of the corresponding null-space. For the space L^2, if $N(\mathfrak{A})$ is finite then $\mathfrak{A}$ may be represented as an integral operator.

J. V. Wehausen (Providence, R. I.).

Kaplansky, Irving. Completely continuous normal operators with property L. Pacific J. Math. **3**, 721–724 (1953).

This paper sets out to generalize the results of Motzkin and others about matrices with property L to completely continuous linear operators on complex Hilbert space. For two such operators, A and B, A is said to have a property L with B if every pair of complex α and ν for which $\nu=\alpha\lambda_i+\mu_i$ for at least k different i's has ν as an eigenvalue of $\alpha A+B$ with multiplicity at least k, where $\{\lambda_i\}$ and $\{\mu_i\}$ are the point spectrum of A and B respectively counted with proper multiplicity. The author proves that such A and B, if Hermitian, must commute, a complete generalization of the finite-dimensional case. If A and B are only normal, he gets this result only in case all $\mu_i\neq 0$. *F. H. Brownell.*

***Nakano, Hidegorô. Spectral theory in the Hilbert space.** Japan Society for the Promotion of Science, Tokyo, 1953. iv+300 pp. $3.00.

This book is a highly individualistic account of the known facts (and some of their generalizations) concerning operators on Hilbert spaces. The author's rugged individualism manifests itself not only in his occasionally novel approach, but also in his insistence on his personal and rather unusual terminology. Thus, for example, the cardinal number of a set is called its "density", by an "ideal" in a Boolean ring the author means a dual ideal, a spectral measure becomes a "spectrality", and the interior of a set in a topological space is called its "opener". Such terminology, when combined with a difficult expository style and frequent violations of idiomatic English usage, results in a book that is very hard to read.

The book consists of six chapters: I. Hilbert spaces. II. Spectral theory. III. Dilatator analysis. IV. Normal operators. V. Unitary invariants. VI. Ergodic theorems. (A "dilatator", by the way, is a closed operator with dense domain satisfying a commutativity condition with respect to a certain Boolean ring of projections.) The climax of the book is reached in chapter V, where the author presents his version of the multiplicity theory. Motivated by that theory, after getting the usual technical preliminaries out of the way, the author formulates all his definitions and proves all his theorems relative to a fixed Boolean ring of projections, and, in this connection, he makes systematic use of the associated Stone space. These two aspects of his presentation (realization and dualization) constitute the principal novelty in the book. A specialist in the field might profit from studying the book in detail and observing the systematic application of techniques that are known but not yet trite.

P. R. Halmos (Chicago, Ill.).

Herstein, I. N. Une note sur un article de M. Tumuraru. Portugaliae Math. **12**, 113–114 (1953).

Let R be an algebra over the complex numbers with an adjunction operation $*$ having the property that $aa^*=0$ implies $a=0$. If the product $x\circ y=\frac{1}{2}(xy+yx)$ is associative on the self-adjoint elements of R, then R is commutative. This extends a result of Turumaru [Kôdai Math. Sem. Rep. **1951**, 51; these Rev. **13**, 565, 1140] (whose name is not correctly given in the title of the paper under review). The proof is short and purely algebraic. The latter in contrast to Turumaru's, and the technique yields an extension of a theorem of Ancochea [Ann. of Math. (2) **48**, 147–153 (1947); these Rev. **8**, 310]. *I. E. Segal* (New York, N. Y.).

Herstein, I. N. A note on a commutativity theorem. Kôdai Math. Sem. Rep. **1953**, 119–120 (1953).

This is identical with the note reviewed above.

I. E. Segal (New York, N. Y.).

Wermer, John. On algebras of continuous functions. Proc. Amer. Math. Soc. **4**, 866–869 (1953).

Let C denote the Banach algebra of continuous functions on the unit circle: $|\lambda|=1$. Theorem 2. Let B be any closed subalgebra of C which contains one function $\varphi(\lambda)$ such that

$\varphi(\lambda_1)=\varphi(\lambda_2)$ implies $\lambda_1=\lambda_2$. If B' is a closed subalgebra of C which includes B, then $B'=B$ or $B'=C$. This is a consequence of the special case (theorem 1) where B is the set of functions in C which are boundary values of functions analytic in $|z|<1$ and continuous in $|z|\leqq 1$ ($\varphi(\lambda)=\lambda$). The proof of theorem 1 is based upon the representation of an arbitrary linear functional on C as an integral with respect to a complex measure on the circle, and the representation of every measure μ which satisfies the condition $\int_{|\lambda|=1}\lambda^n d\mu(\lambda)=0$, $n\geqq 0$, in terms of the boundary values of a function analytic in $|z|<1$. Theorem 1 is the solution of a problem of Leĭbenzon [Uspehi Matem. Nauk (N.S.) **7**, no. 4(50), 163–164 (1952); these Rev. **14**, 386], and has as a corollary a theorem of Rudin [Duke Math. J. **20**, 449–457 (1953); these Rev. **15**, 21]. *M. Jerison.*

Sunouchi, Haruo. The irreducible decompositions of the maximal Hilbert algebras of the finite class. Tôhoku Math. J. (2) **4**, 207–215 (1952).

A proof that the algebras of left and right multiplications of a maximal Hilbert algebra (in the sense of Nakano) decompose into direct sums of factors. The paper follows very closely two papers of Godement [Ann. of Math. (2) **53**, 68–124 (1951); J. Math. Pures Appl. (9) **30**, 1–110 (1951); these Rev. **12**, 421; **13**, 12] in applying standard techniques. *W. Ambrose* (Cambridge, Mass.).

Godement, Roger. Théorie des caractères. I. Algèbres unitaires. Ann. of Math. (2) **59**, 47–62 (1954).

A unitary algebra $\mathbf{A}$ is a *-algebra over the complex numbers equipped with an inner product under which it is a pre-Hilbert space and such that $(xy,z)=(y,x^*z)$, $(x,y)=(y^*,x^*)$, $y\rightarrow xy$ is continuous for each x and the products xy are dense in $\mathbf{A}$. Let $\mathfrak{H}$ be the completion of $\mathbf{A}$ and let S and U_x denote the unique continuous extensions of $y\rightarrow y^*$ and $y\rightarrow xy$. Let $V_x=SU_x^*S$ and let R^s and R^d denote the (von Neumann-Murray) operator rings generated by the U_x and V_y respectively. Then (Theorem 1) R^s is the commuting algebra of R^d and vice versa. Now let M be any operator ring and let M^+ denote the set of all non-negative self-adjoint operators in M. By a trace on M the author means a function Tr from M^+ to $[0,\infty]$ such that $\mathrm{Tr}(UHU^{-1})=\mathrm{Tr}(H)$ for all unitary $U\in M$ and $\mathrm{Tr}(\sum H_\alpha)=\sum \mathrm{Tr}(H_\alpha)$ whenever $\sum H_\alpha$ is strongly convergent. Given a trace Tr on M, let M_2 be the set of all $A\in M$ with $\mathrm{Tr}(A^*A)<\infty$. Then (Theorem 2) M_2 is a unitary algebra with respect to the inner product $(A,B)=\mathrm{Tr}(B^*A)$ provided that Tr is "faithful" in the sense that $\mathrm{Tr}(H)=0$ implies $H=0$. Theorem 3 is a partial converse of Theorem 2. It asserts that if $\mathbf{A}$ is any unitary algebra then there exists a unique faithful trace Tr in R^s such that (a) $\mathrm{Tr}(A^*A)<\infty$ if and only if there exists a "bounded" $a\in\mathfrak{H}$ such that $A=U_a$ and (b) $\mathrm{Tr}(U_aU_b^*)=(a,b)$ for all "bounded" a and b in $\mathfrak{H}$. Here a is said to be bounded if $V_x(a)$ is bounded as a function of x and U_a is the unique bounded extension of this operator to $\mathfrak{H}$. The mapping $x\rightarrow U_x$ thus provides a mapping of $\mathbf{A}$ into the unitary algebra associated with a trace in an operator ring. Theorem 4 deals with the subalgebras of the unitary algebra defined by M and Tr in Theorem 2. It is shown that if M is a factor not of type III and A is a self-adjoint subalgebra of M_2 which is strongly dense in M, then A is itself a unitary algebra application of Theorem 3 to which leads back to the original M and Tr. A generalization of Theorem 4 in which M is not required to be a factor is conjectured and in a note added in proof it is reported that this conjecture has been proved by Dixmier.

The correspondence between traces in operator algebras on the one hand and unitary algebras on the other which this paper elucidates has been discussed from a slightly different point of view by I. E. Segal in a paper [Ann. of Math. (2) **57**, 401–457 (1953), Part 5; these Rev. **14**, 991] which did not appear until several months after the present paper had been submitted. Segal uses a "gage", i.e. a measure defined on projections instead of a trace and uses a formally different definition of unitary algebra.

G. W. Mackey (Cambridge, Mass.).

Godement, Roger. Théorie des caractères. II. Définition et propriétés générales des caractères. Ann. of Math. (2) **59**, 63–85 (1954).

The chief purpose of this paper is to introduce a general notion of character for representations of unimodular locally compact groups and study some of its properties. In an earlier work [J. Math. Pures Appl. (9) **30**, 1–110 (1951); these Rev. **13**, 12] the author studied a definition which was adequate for the special class of groups there considered but turned out to be inadequate more generally. The new definition is such as to allow recovery of a good part of the classical one-to-one correspondence between characters and irreducible representations provided that attention is restricted to representations which are normal in a certain sense. Of course irreducible representations must be replaced by factor representations.

Let G be a locally compact unimodular group and let $M(G)$ be the *-algebra of all Radon set functions of bounded variation on G topologized with the weakest topology under which $\mu\rightarrow\int f(x)d\mu(x)$ is continuous for all bounded continuous functions on G. A group algebra $\mathfrak{a}$ for G is any dense self-adjoint subalgebra of $M(G)$ which is invariant under translation. A trace is a complex-valued function σ defined on $\mathfrak{a}\times\mathfrak{a}$, where $\mathfrak{a}$ is some group algebra for G, such that: (a) $\sigma(\alpha,\beta)$ is linear in α and antilinear in β, (b) $\sigma(\alpha,\alpha)\geqq 0$, (c) $\sigma(\alpha_s,\beta_s)=\sigma(\alpha,\beta)=\sigma(\beta^*,\alpha^*)$ where α_s denotes the translate of α by s, (d) $\sigma(\beta_s,\gamma)$ is continuous in s and $\sigma(\alpha\beta,\gamma)=\int\sigma(\beta_s,\gamma)d\alpha(s)$. The domain $\mathfrak{a}\times\mathfrak{a}$ of σ will in general depend upon σ. Every trace σ defines a unitary algebra in the sense of part I of this series [see the immediately preceding review] on the quotient of $\mathfrak{a}$ by the set N of all α with $\sigma(\alpha,\alpha)=0$ (Theorem 1). Passing to the corresponding trace in the ring R^s associated with this unitary algebra in paper I and back to $M(G)$ one obtains an extension of σ to a larger group algebra. If this extension is in fact σ itself, then σ is said to be a maximal trace. Every trace has a unique maximal extension (part of Theorem 3). Finally a character of G is a maximal trace such that the R^s associated with the corresponding unitary algebra is a factor.

A factor representation $x\rightarrow T_x$ is normal if there exists a group algebra $\mathfrak{a}$ such that $T_\alpha T^*_\alpha$ has a finite relative trace for all α in $\mathfrak{a}$. Every such representation has associated with it a character determined up to a positive multiplicative constant. Every character arises in this way from a normal factor representation and two representations T' and T'' define proportional characters if and only if there exists an isomorphism φ of the corresponding factors such that $\varphi(T_s')=T_s''$ for all $s\in G$ (Theorem 6). An irreducible representation is determined up to unitary equivalence by its character (part of Theorem 7). Characters may be characterized amongst maximal traces by an extremal property (Theorem 5). If G has a compact subgroup K such that every irreducible representation of G has only finite multiplicities when restricted to K, then every factor repre-

sentation of G is normal and of type I and there exists a single group algebra on which all characters are defined (Theorem 8).

Existence and decomposition theorems are promised in a subsequent article. *G. W. Mackey.*

Tagamlicki, Ya. Investigation of vectors which are irreducible relative to certain cones. Bŭlgar. Akad. Nauk. Izvestiya Mat. Inst. 1, 57–68 (1953). (Bulgarian. Russian summary)

Irreducible elements of cones in inner product spaces, which have been utilized by the author in a previous communication [Annuaire [Godišnik] Fac. Sci. Phys. Math., Univ. Sofia, Livre 1, Partie II. 47, 85–107 (1952); these Rev. 15, 135] are here applied to a special case of a construction due to M. G. Kreĭn [Uspehi Matem. Nauk (N.S.) 6, no. 4(44), 3–120 (1951); these Rev. 13, 445]. Let S_n be the cone in R^{n+1} consisting of all $\{a_0, a_1, \cdots, a_n\}$ such that $\sum_{i=0}^n a_i\alpha_i \geqq 0$ for all $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ such that $\sum_{i=0}^n \alpha_i t^i \geqq 0$ for all $t \geqq 0$. It is proved that the irreducible elements of S_n are (1) all vectors $\{0, 0, \cdots, 0, a_n\}$ with $a_n > 0$, (2) all vectors $\{a_0, a_0 q, \cdots, a_0 q^n\}$ with $q > 0$. The author's theorem on irreducible elements [loc. cit.] enables him to show that the elements of S_n all have the representation

$$\left\{c\delta_{rn} + \int_0^\infty t^r d\alpha(t)\right\}_{r=0}^n$$

for $c \geqq 0$ and α a monotone increasing function on $[0, \infty]$ with $\alpha(\infty) < \infty$. A passage to the limit as $n \to \infty$ is used to solve the Stieltjes moment problem, although the details are not given. A similar argument with a different cone leads to a solution of the Hamburger moment problem.

E. Hewitt (Seattle, Wash.).

Nakano, Hidegoro. Concave modulars. J. Math. Soc. Japan 5, 29–49 (1953).

The author studies an abstraction R of the space L^p, $0 < p < 1$. R is defined as a σ-conditionally complete vector lattice with a non-negative function $m(x)$ designed to generalize $\int |x(t)|^p dt$ in L^p, $0 < p < 1$. The postulates on $m(x)$ are: $m(x) \leqq m(y)$ whenever $|x| \leqq |y|$; $m(x+y) = m(x) + m(y)$ whenever $x \cap y = 0$; for every fixed x the function $m(tx)$ is concave in $t \geqq 0$ with $m(tx) \to 0$ as $t \to 0$; if $0 \leqq x_1 \leqq x_2 \leqq \cdots$ with $m(x_n)$ bounded, then there should exist an x with $x = \sup x_n$ and $m(x) = \sup m(x_n)$; $m(x) > 0$ for all $x \neq 0$. Among other results the author shows that: R has the property that for every system $0 < x_\lambda$ there is a countable subset x_n with $\bigwedge_\lambda x_\lambda = \bigwedge_n x_n$, and hence R is necessarily conditionally complete; R has a unique decomposition into orthogonal normal manifolds F and S such that, as $t \to \infty$, $\lim m(tx)/t > 0$ for every $x \neq 0$ in F and $= 0$ for every x in S; if $R = S$ and R has no discrete element then the only bounded linear functional on R is the zero functional, this last result having been obtained previously by M. M. Day [Bull. Amer. Math. Soc. 46, 816–823 (1940); these Rev. 2, 102, 419] and G. Sirvint [C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 119–122 (1940); these Rev. 2, 180] for the particular case of L^p, $0 < p < 1$.

I. Halperin (Kingston, Ont.).

Parasyuk, O. S. Flows of horocycles on surfaces of constant negative curvature. Uspehi Matem. Nauk (N.S.) 8, no. 3(55), 125–126 (1953). (Russian)

This paper uses the definitions and methods introduced by I. M. Gelfand and S. V. Fomin [same journal (N.S.) 7, no. 1(47), 118–137 (1952); these Rev. 14, 660]. The manifold of directed line elements on a surface F of constant negative curvature is identified with the quotient group G/D where G is the group of all real 2×2 matrices with determinant 1 and D is a discrete group isomorphic to the fundamental group of F. The flow is defined by multiplication by $g_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. In this flow each line element moves along and always perpendicular to a horocycle of the Lobačevskiĭ plane which covers F. Theorem 1. A horocycle flow on a surface of constant negative curvature has a Lebesgue spectrum. Theorem 2. If the surface has finite area, the horocycle flow is ergodic and strongly mixing. The proof of theorem 1 is sketched. Theorem 2 follows from theorem 1. *Y. N. Dowker* (London).

Calculus of Variations

Sigalov, A. G. Two-dimensional problems of the calculus of variations in nonparametric form. Trudy Moskov. Mat. Obšč. 2, 201–233 (1953). (Russian)

The principal result of this paper, which is gotten by combining Theorems A and B below, was announced by Sigalov in a somewhat different form in a note in Doklady Akad. Nauk SSSR (N.S.) 73, 891–894 (1950); these Rev. 12, 268.

He is concerned with integrals of the form

$$(f, D, F) = \iint_D F(x, y, z, p, q)\,dxdy.$$

Here D is a bounded region, $z = f(x, y)$, $p = f_x$, and $q = f_y$. The boundary values of $f(x, y)$ are prescribed and continuous, and f is required to belong to the class L_α ($\alpha \geqq 1$) of continuous functions satisfying the two following conditions: (i) $f(x, y)$ is AC in x for almost all y and vice versa; (ii) $\iint_D (f_x^2 + f_y^2)^{\alpha/2} dxdy < \infty$. The boundary D' is "uniformly regular", that is, the diameters of its components are bounded from zero.

The author first states a theorem "in the bounded domain". Theorem A: Let F, F_p, and F_q be continuous, the Weierstrassian nonnegative, and suppose that for some positive m and for (p^2+q^2) large enough $F/(p^2+q^2) \geqq m$. Suppose finally that the "approximation theorem" holds for F (see below). Then if there is an f in L_α taking on the prescribed boundary values, satisfying $|f(x, y)| \leqq z_0$ on $\bar{D}$, and yielding a finite value for (f, D, F), some f_0 in L_α minimizes (f, D, F) among all $f \in L_\alpha$ with the bound z_0 and the prescribed boundary values.

The proof of Theorem A rests mainly on the author's regions of redefinition. Because F satisfies the approximation theorem, one can start with a minimizing sequence of piecewise linear functions. A region G with $\bar{G} \subset D$ is fundamental if, for some c, G is a component of the set $\{f(x, y) < c\}$. G is a region of redefinition for the function f if G is fundamental and $(f, G, F) > (c, G, F)$. Sigalov has proved elsewhere [Uspehi Matem. Nauk (N.S.) 6, no. 2(42), 16–101 (1951); see also Amer. Math. Soc. Translation no. 83 (1953), the theorem on p. 67; these Rev. 13, 257; 14, 769] that a piecewise linear minimizing sequence can be replaced by another, still piecewise linear, whose functions have no regions of redefinition. Hence for every fundamental region G_c associated with the value c of the function f_n we have $(f_n, G_c, F) \leqq (c, G_c, F)$. Using this fact, the growth condition on F, and the compactness of the (x, y, z)-space, the author

proves that the average of the gradient of f_n over a fundamental region for f_n is bounded independently of n and of the region in question. From this he is able to derive a simple bound for the difference of the oscillations on a fundamental region G and on its boundary. From the growth condition on F he proves the uniform boundedness of the Dirichlet integrals. This gives him a family of Young $\epsilon-\delta$ gratings. With the gratings, and the estimate of the difference of the oscillations, he is able to prove equicontinuity without much further difficulty. Arzelà's theorem and well-known lower semicontinuity theorems finish it off.

The approximation theorem mentioned above holds for F whenever given a function $f \in L_\alpha$ there is a piecewise-linear function f_1 close to f in the Fréchet sense and approximating the integral (f, D, F). Sigalov publishes in this connection a theorem communicated to him by V. I. Plotnikov. This theorem is as follows: Let F be defined and continuous for $(x, y) \in \bar{D}$ and $|z| \leqq z_0$, p and q arbitrary. Suppose there are constants L_1 and L_2 and a continuous convex function $\phi(p, q)$ such that $L_1\phi(p, q) \leqq F(x, y, z, p, q) \leqq L_2\phi(p, q)$. Then the approximation theorem holds for F. The proof is straightforward and based on h-averages.

In order to remove the restriction $|z| \leqq z_0$ which appeared in Theorem A, the author proves a theorem giving conditions under which there exists a bounded minimizing sequence. Define $\phi(z) = \inf \{F/(p^2+q^2)^{\frac{1}{2}}\}$, the inf being taken, for some fixed number L, over all (x, y, p, q) satisfying $(x, y) \in D$ and $p^2+q^2 \geqq L$. Define further $\psi(z) = \sup F(x, y, z, 0, 0)$, the sup being taken over all $(x, y) \in D$.

Theorem B is as follows: Let F be continuous for $(x, y) \in \bar{D}$ and for all z, p, and q. Let F satisfy the approximation theorem and suppose that $F \geqq 0$. Suppose that some function f in L_α takes on the prescribed boundary values and satisfies $(f, D, F) < \infty$. Finally suppose that for any $N > 0$ there exist N_1 and N_2, with N_1 larger than the maximum absolute value of the boundary values prescribed for f, such that any collection $(a_1, b_1), \cdots, (a_n, b_n)$ of nonoverlapping intervals satisfying $a_jb_j > 0$, $|a_j| > N_1$, $|b_j| > N_1$ $(j = 1, \cdots, n)$ and $\sum_j |b_j - a_j| > N_2$ must also satisfy

$$\sum_j \int_{a_j}^{b_j} \phi\psi^{-\frac{1}{2}} dz > N.$$

Then there exists a bounded minimizing sequence.

To prove Theorem B Sigalov begins with an arbitrary minimizing sequence of piecewise linear functions. After disposing of a trivial case, he applies again the theorem on regions of redefinition. The "redefined" minimizing sequence is the bounded sequence desired. The average-gradient method is again applied, this time to a macroscopic situation. The strange-appearing final condition in Theorem B arises naturally on breaking of the interval of variation of the quantity c.

On combining Theorems A and B the author gets a general existence theorem in the unbounded domain. He states a number of examples.

The final sections of the paper are concerned with finding conditions under which the solutions of these problems satisfy Lipschitz conditions. In order to do this he generalizes a number of ideas of Shiffman [Ann. of Math. (2) **48**, 274–284 (1947); these Rev. **9**, 45] to functions which are not Lipschitzian. Again he makes use of the fundamental regions. The motivation for this work is the theorem of Morrey [Trans. Amer. Math. Soc. **43**, 126–166 (1938)] that if the solution is Lipschitzian then it possesses the same functional properties as the integrand.

The reviewer admits himself confused by some apparently conflicting statements. Consider the condition: $F/[(p^2+q^2)^{\alpha/2}] \geqq m$ for $p^2+q^2 \geqq L$. In his earlier announcement of these results the author stated that they were valid for $\alpha > 1$. In the introduction to this paper he states that the theorem we have called Theorem A is proved for $\alpha \geqq 1$. In the statement of Theorem A the condition is stated for $\alpha \geqq 2$, i.e. in the form found in the third paragraph of this review. Nevertheless it appears (on pp. 218–219 of his paper) that the results are applied to examples in which $1 \leqq \alpha < 2$. There is no question in the reviewer's mind as to the internal consistency of the paper if the introduction and examples are not taken into account. *J. M. Danskin.*

Sigalov, A. G. Two-dimensional problems of the calculus of variations in nonparametric form transformed into parametric form. Doklady Akad. Nauk SSSR (N.S.) **93**, 405–408 (1953). (Russian)

In two earlier papers [same Doklady (N.S.) **73**, 891–894 (1950); **81**, 741–744 (1951); these Rev. **12**, 268; **13**, 758] the author announced a theorem asserting the existence of an absolute minimum for the integral $\iint_D F(x, y, z, p, q)dxdy$ when the admissible functions take on given values on the boundary and the integrand satisfies the condition $F \geqq m(1+p^2+q^2)^{\alpha/2}$ when $p^2+q^2 \geqq L^2$, $m > 0$ and $\alpha > 1$, and certain other conditions are satisfied. In the paper reviewed above he gave the details on which the first two papers were based. In the introduction to this last paper he again asserted this theorem for $\alpha > 1$. However, the proof given in the body of the paper holds only in the case $\alpha \geqq 2$.

The author's object in the present note is to call attention to this error, which obviously weakens his results very much (for $\alpha > 2$ Tonelli proved approximately this theorem twenty years ago [Acta Math. **53**, 325–346 (1929); Ann. Scuola Norm. Super. Pisa (2) **2**, 89–130 (1933); see also Morrey, Bull. Amer. Math. Soc. **46**, 439–458 (1940); these Rev. **2**, 60] and Morrey extended it to $\alpha = 2$ [Univ. California Publ. Math. (N.S.) **1**, 1–130 (1943); these Rev. **6**, 180]), and then presents some results which partially restore the situation in the range $1 < \alpha < 2$.

His solutions are gotten by converting the problem to parametric form and accepting parametric surfaces as solutions of the nonparametric problem. By generalizing somewhat the class of admissible nonparametric functions $z = f(x, y)$, he then obtains a weakened form of his previous assertion. Proofs and details are apparently reserved for a later paper. *J. M. Danskin* (Washington, D. C.).

Theory of Probability

Dufresne, Pierre. Problèmes de dépouillements. C. R. Acad. Sci. Paris **238**, 42–44 (1954).

A finite population consists of elements pertaining to two (or more) classes. The composition of the population is assumed to be known. Elements are drawn at random until the population is exhausted. The author determines probabilities that at every stage of the process of drawing the composition of the sample extracted should satisfy certain conditions. No proofs are given. *E. Lukacs.*

Spitz, J. C. Matching in psychology. Statistica, Rijswijk **7**, 23–40 (1953). (Dutch. English summary)

A detailed description of the preparation of a table of the probability of a or more hits in matching n independent

sets of three objects ($n=1(1)30$; 3D). A practical approximation to this distribution would be the Poisson exponential with mean n. For $n=10$ only three entries out of 22 in this table differ by as much as three units from the corresponding Poisson figure. *H. L. Seal* (New York, N. Y.).

Hida, Takeyuki. On some asymptotic properties of Poisson process. Nagoya Math. J. 6, 29–36 (1953).

Let $X(t, \omega)$ ($\omega \in \Omega$, $0 \leqq t < \infty$) be the Poisson process and $L_m(\omega) = t_{m+1}(\omega) - t_m(\omega)$ where $t_m(\omega) = \min\{\tau, X(\tau, \omega) = m\}$, $m = 0, 1, 2, \cdots$. Then $L_m(\omega)$ are mutually independent random variables with common distribution $F(l) = 1 - e^{-\lambda l}$ if $l \geqq 0$ and 0 otherwise. Let $M_n(\omega)$ and $m_n(\omega)$ be respectively the maximum and minimum of the first n L_m's. The author proves that, for $0<\alpha<1$, $\Pr\{\lim\inf_{n\to\infty} (\lambda M_n/\alpha \log n) \geqq 1\} = 1$, and if $\beta>1$, then $\Pr\{\lim\sup_{n\to\infty} (\lambda m_n/\beta n^{-1} \log n) \geqq 1\} = 0$, where λ is the parameter of the process. Let

$$Z_n(\omega) = \sum_{j=0}^{n-1} L_j(\omega)/M_n(\omega).$$

Then it is proved that the mean and standard deviation of Z_n are of order $n/\log n$. *J. L. Snell* (Princeton, N. J.).

Kanellos, S. G. On a theorem of N. Kritikos. Bull. Soc. Math. Grèce 27, 111–114 (1953). (Greek)

The author gives a short proof of a theorem of Kritikos [same Bull. 24, 111–118 (1949); these Rev. 11, 654], and uses Kolmogorov's three-series theorem to extend Kritikos' result to a more general averaging process.

J. M. Danskin (Washington, D. C.).

Kanellos, S. G. On the "comparative frequency" of an event. Bull. Soc. Math. Grèce 27, 25–67 (1953). (Greek. English summary) See the note on p. 1140

This is a long elementary expository paper, proving among other things Doob's theorem [Ann. of Math. (2) 37, 363–367 (1936)] on the impossibility of a successful gambling system. *J. M. Danskin* (Washington, D. C.).

Darling, D. A. On a class of problems related to the random division of an interval. Ann. Math. Statistics 24, 239–253 (1953).

Let $X_1, \cdots, X_n$ be independent random variables each distributed uniformly over $(0, 1)$, and let $Y_0, Y_1, \cdots, Y_n$ be the respective lengths of the $n+1$ segments into which the interval is divided by the $\{X_i\}$. The main result obtained is a contour integral for the characteristic function of $W_n = \sum_{j=0}^{n} h_j(Y_j)$ for arbitrary functions $h_j(x)$. The characteristic function of the Y_j's is also derived. Statisticians have considered various functions of the form $h_j(x) = h(x)$ (e.g., $h(x) = x^\alpha$, $\alpha > 0$) in connection with non-parametric goodness-of-fit tests and in connection with the Poisson stochastic process. The author gives the distribution of W_n for these and other functions in an asymptotic form generally and in closed form in some cases. Included among the results are asymptotic distributions associated with the number of Y_j's satisfying inequalities of the form $\alpha < Y_j < \beta$ ($j = 0, 1, \cdots, n$). [In Theorem 2.1 replace "real-valued" by "complex-valued."] *D. F. Votaw, Jr.*

Erdös, P., and Hunt, G. A. Changes of sign of sums of random variables. Pacific J. Math. 3, 673–687 (1953).

Considérant une suite indéfinie de variables aléatoires $X_1, X_2, \cdots, X_k, \cdots$, mutuellement indépendantes et admettant la même fonction de répartition $F(x)$ continue et symétrique par rapport à 0, et posant $S_k = X_1 + X_2 + \cdots + X_k$, les auteurs établissent des propriétés asymptotiques (pour $n \to +\infty$) du nombre N_n des changements de signe apparaissant dans la suite $S_1, S_2, \cdots, S_n$, et de son espérance mathématique; par exemple, il y a une probabilité 1 pour que $\lim\inf_{n\to+\infty} N_n/\log n \geqq \frac{1}{2}$. Ces propriétés sont indépendantes de $F(x)$ et s'étendent partiellement au cas où $F(x)$ présente des discontinuités. *R. Fortet* (Paris).

Andersen, Erik Sparre. On the fluctuations of sums of random variables. Math. Scand. 1, 263–285 (1953).

Let $X_1, X_2, \cdots$ be random variables such that (a) the joint distribution of any number of them is symmetrical in all the variables or more specifically (b) the X's are independent, identically distributed. Let $S_i = \sum_{j=1}^{i} X_j$; L_n be the smallest $i \geqq 0$ for which $S_i = \max(0, S_1, \cdots, S_n)$; M_n be the largest $i \leqq n$ for which $S_i = \min(0, S_1, \cdots, S_n)$; N_n be the number of positive terms in $S_1, \cdots, S_n$; and N_n^* be the number of j, $1 \leqq j \leqq n$, for which $S_j j^{-1} > S_{n+1}(n+1)^{-1}$. Let K_n be any one of L_n, M_n and N_n. Assume (a). If C is an event symmetrical with respect to $X_1, \cdots, X_{n+1}$, then

$$\Pr(K_n = m \mid (S_{n+1} = 0)C) = (n+1)^{-1}$$

for $0 \leqq m \leqq n$ if and only if $\Pr((S_i = S_{n+1} = 0)C) = 0$ for $1 \leqq i \leqq n$. Corollaries: (1) If the joint distribution of $X_1, \cdots, X_{n+1}$ is absolutely continuous and if C is as before with $\Pr(C) > 0$, then $\Pr(N_n^* = m \mid C) = (n+1)^{-1}$ for $0 \leqq m \leqq n$; (2) if the X's assume only integer values and if C is as in (1) and $C \subset (S_{n+1} = 0)$, then $\Pr(K_n = m \mid C) = (n+1)^{-1}$ for $0 \leqq m \leqq n$. The prototype of such results asserting strict uniformity (among all possible values) is one by Chung and Feller [Proc. Nat. Acad. Sci. U. S. A. 35, 605–608 (1949); these Rev. 11, 444] dealing with the coin-tossing case. The present results do not apparently cover this, but are reformulations carrying greater generality. Now assume (b). Let $c_n = \Pr(S_n = 0)$, $f_n = \Pr(\bigcap_{i=1}^{n-1}(S_i < 0)(S_n = 0))$, $u_n = \Pr((L_n = 0)(S_n = 0))$. Various relations between these probabilities are obtained, in particular,

$$\Pr((K_n = m)(S_{n+1} = 0)) = \sum_{k=m}^{n} f_{k+1} u_{n-k}.$$

From this formula the conditional distribution of K_n under the hypothesis $S_{n+1} = 0$ is calculated. If, e.g., it is assumed that c_n is of the true order n^{-a} for some positive a, then the said distribution is asymptotically uniform with a sharp estimate of the error term, improving a result by M. Lipshutz [Proc. Amer. Math. Soc. 3, 659–670 (1952); these Rev. 14, 662]. The exact algebraic results in this paper are obtained by transforming events by suitable permutations of the X's. On p. 267, l. 18 read $\sum\sum$ for $\bigcap\bigcap$; on p. 268, l. 2 and 3 are meaningless and need rewording.

K. L. Chung (Syracuse, N. Y.)

Finkel'štein, B. V. On the limiting distributions of the extreme terms of a variational series of a two-dimensional random quantity. Doklady Akad. Nauk SSSR (N.S.) 91, 209–211 (1953). (Russian)

Let $(x_1, y_1), \cdots, (x_n, y_n)$ be independent 2-vectors with a common distribution. Let $\xi_1^{(n)} \leqq \xi_2^{(n)} \leqq \cdots \leqq \xi_n^{(n)}$ and $\eta_1^{(n)} \leqq \eta_2^{(n)} \leqq \cdots \leqq \eta_n^{(n)}$ be re-orderings of the x and y resp. Several theorems are stated without proof about the limit distribution of the minimal pair $(\xi_1^{(n)}, \eta_1^{(n)})$, ending with remarks on same for more general $(\xi_k^{(n)}, \eta_m^{(n)})$. They are long and must await details. *K. L. Chung.*

Gihman, I. I. Some limit theorems for conditional distributions. Doklady Akad. Nauk SSSR (N.S.) 91, 1003–1006 (1953). (Russian)

Let $\xi_1, \cdots, \xi_n, \cdots$ be independent, identically distributed random variables with $M\xi_k=0$, $M\xi_k^2=\sigma_k^2<\infty$ and suppose that the common distribution either has a density function of bounded variation or is of the lattice type. Let $\eta_k=\sigma^{-1}n^{-1/2}\sum_{r=1}^k\xi_r$, $M(n)=\max_{0\leq k\leq n}\eta_k$, $m(n)=-\min_{0\leq k\leq n}\eta_k$, $\zeta_k=\eta_k-(k/n)\eta_n$, $M'(n)=\max_{1\leq k\leq n}\zeta_k$, $m'(n)=-\min_{1\leq k\leq n}\zeta_k$, v_n=no. of positive terms in ζ_k, $1\leq k\leq n$. (η_0 is not defined but is presumably 0.) If $n\rightarrow\infty$, $z_n\rightarrow z$, then the limit conditional distribution of $\{m(n), M(n)\}$, that of $\{m'(n), M'(n)\}$ and that of v_n/n, all three under the condition that $\eta_n=z_n$, are given. The last is uniform. The method used is that of reduction to partial differential equations with the help of upper and lower functions as set forth in Khintchine's "Asymptotische Gesetze der Wahrscheinlichkeitsrechnung" [Springer, Berlin, 1933]. No details; some references to the literature seem misplaced. *K. L. Chung.*

HAVE

★**Doob, J. L. Stochastic processes.** John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, London, 1953. viii+654 pp. $10.00.

In this valuable book the author defines a stochastic process as "any process running along in time and controlled by probabilistic laws", or, more precisely, as "any family of random variables $\{x_t: t \in T\}$" where "a random variable is . . . simply a measurable function". He observes that "probability theory is simply a branch of measure theory, with its own special emphasis and field of application", and adheres uncompromisingly to this point of view throughout the book. It follows that one really is working all the time with genuinely probabilistic arguments, instead of performing analytical tricks with distribution functions, but a thorough familiarity with measure theory has to be assumed. A Supplement of 23 pages summarises what is needed, but it is rather an examination which must be passed by a candidate wishing to be admitted to Chapter I, than a course in itself. Two of the topics discussed in the Supplement deserve special mention.

(1) Construction of measures on product spaces. Doob's treatment of this is based on a paper by C. T. Ionescu Tulcea [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 208–211 (1950); these Rev. 12, 85] which constructs a measure on the cartesian product of countably many replicates of an abstract space X in terms of functions q_n on $X^n\times\mathfrak{F}_X$ to the real line ($\mathfrak{F}_X$ is a Borel field of subsets of X) which determine generalised Baire functions on X^n and probability-measures on $(X, \mathfrak{F}_X)$. When $q_n(x_1, \cdots, x_n; A)$ is independent of $x_1, \cdots, x_{n-1}$ we have a "Markov-chain" measure; when it depends on none of the x's we have "direct product" measure. The theorem is used to derive (i) the existence of the direct product measure on X^T (where T is arbitrary) and (ii) Kolmogorov's theorem concerning the existence of measures on R^T (where R is the real line) (the last requires the author's theory of conditional distributions discussed below, and here the reader is left to fill in the gaps of a highly condensed argument). This construction of Markov-chain measure would run into difficulties when X is abstract and T is not the set of positive integers. If T is "all integers" then "backwards" transition-probabilities must be given at the outset, and if T is R then "two-sided" transition probabilities are required.

(2) Representation theory. (A thorough grasp of this is essential to an understanding of the author's point of view.) $\mathbf{T}$ is called a single-valued measure-preserving point transformation of one probability-space $(\Omega, \mathfrak{F}, q)$ into another $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{q})$ when $\mathbf{T}$ is a single valued map of Ω into $\tilde{\Omega}$, $\mathbf{T}^{-1}$ maps $\tilde{\mathfrak{F}}$ onto $\mathfrak{F}$ and $q(\mathbf{T}^{-1}\tilde{\Lambda})=\tilde{q}(\tilde{\Lambda})$ for all $\tilde{\Lambda} \in \tilde{\mathfrak{F}}$. Such transformations are discussed in detail, and an important theorem is proved, the essence of which is that the corresponding mapping from random variables on $(\tilde{\Omega}, \tilde{\mathfrak{F}})$ to random variables on $(\Omega, \mathfrak{F})$ is "onto". In the constantly recurring application, $(\Omega, \mathfrak{A}, q)$ is the basic probability-space, $\{x_t: t \in T\}$ is a stochastic process (S.P.) thereon, $\mathfrak{F}$ is $\mathfrak{B}(x_t: t \in T)$ (the smallest Borel field with respect to which all the x's are measurable), $\tilde{\Omega}$ is the function-space R^T, $\tilde{x}_t$ is the tth coordinate function thereon, $\tilde{\mathfrak{F}}$ is the usual Borel field of subsets of R^T, $(\mathbf{T}\omega)(t)=x_t(\omega)$ and $\tilde{q}$ is the probability-measure induced by $\mathbf{T}$. This provides the link between measure-theory and distribution-theory throughout the book. Everywhere in the book the basic probability space is supposed to be complete. A note on completeness in the $\tilde{\Omega}$ space would have been helpful here.

Chapter I ("Introduction and probability background") follows naturally on the Supplement. Some topics (e.g., convergence concepts) are given little space, but conditional expectations, probabilities and distributions receive 22 pages of great importance. If $(\Omega, \mathfrak{A}, \mathbf{P})$ is the basic probability space, then $\mathbf{P}(M|\mathfrak{F})$, the conditional probability of $M \in \mathfrak{A}$ with regard to the Borel field $\mathfrak{F}\subseteq\mathfrak{A}$, is defined as the class of random variables y satisfying: (i) y differs at most on an $\mathfrak{A}$-set of $\mathbf{P}$-measure zero from an $\mathfrak{F}$-measurable function, (ii) y is $\mathbf{P}$-integrable and (iii) $\int_A yd\mathbf{P}=\mathbf{P}(A\cap M)$ for all $A \in \mathfrak{F}$. We have a CD($\mathfrak{F}$) (conditional distribution with regard to $\mathfrak{F}$) when there exists a real-valued function P on $\mathfrak{A}\times\Omega$ such that (i) for each fixed $\omega \in \Omega$ it determines a probability-measure on $(\Omega, \mathfrak{A})$, (ii) for each fixed $M \in \mathfrak{A}$ it determines a function on Ω differing from an $\mathfrak{F}$-measurable function at most on an $\mathfrak{A}$-set of $\mathbf{P}$-measure zero and (iii) $P(M, \omega)$ is a random variable in the class $\mathbf{P}(M|\mathfrak{F})$ for each $M \in \mathfrak{A}$. Notes on the non-existence (in general) of CD($\mathfrak{F}$) are contained in the Appendix. In the text the author proceeds at once to introduce two weaker concepts. Let y_i ($i=1, \cdots, n$) be random variables and let $\mathfrak{B}(y_1, \cdots, y_n)$ be defined as above. A CD($\mathfrak{F}$) "of the y's" differs from a CD($\mathfrak{F}$) only in that M is now restricted to be an element of $\mathfrak{B}(y_1, \cdots, y_n)$.

A CD($\mathfrak{F}$) "of the y's in the wide sense" is the weakest of the three. It (p) is a function on $\mathfrak{B}_n\times\Omega$ ($\mathfrak{B}_n$ being the class of Borel sets in R^n) and has exactly similar properties except that in place of (iii) we require that $p(Y, \omega)$ is to be in the class

$$\mathbf{P}(\{[y_1(\omega'), \cdots, y_n(\omega')] \in Y\}|\mathfrak{F})$$

for every $Y \in \mathfrak{B}_n$. The author proves two important existence theorems. (1) A CD($\mathfrak{F}$) "of the y's in the wide sense" always exists, and one can arrange that $p(Y, \cdot)$ is $\mathfrak{F}$-measurable for all $Y \in \mathfrak{B}_n$. (2) Under a condition (*) a CD($\mathfrak{F}$) "of the y's" always exists, and one can arrange that $P(M, \cdot)$ is $\mathfrak{F}$-measurable for all $M \in \mathfrak{B}(y_1, \cdots, y_n)$.

The condition (*) in (2) is that $\{[y_1(\omega), \cdots, y_n(\omega)]: \omega \in \Omega\}$ is to be a Borel set in R^n. As the author remarks, this condition (*) is a useful one (especially if we are working in the representation space, when it is trivially verified), but it is a shade mysterious. To avoid overburdening the text, remarks about extensions of the theorem (2) are relegated to the Appendix. The reviewer heartily wishes the author had been positively verbose on this topic. If, as is often the case, $\mathfrak{F}=\mathfrak{B}(x_1, \cdots, x_m)$ where the x's are further random variables, then representation theory plus theorem (2) appears

to give a stronger result than theorem (1) and leads to the natural counterpart to the Tulcea theorem mentioned above. In a future edition it would be nice to have this topic treated in extenso.

Chapter I finishes with an excellent survey of characteristic functions, and here there are important new inequalities suggested by earlier work by Wintner.

In Chapter II ("Definition of a S.P.-principal classes") the author gives a solution to the problem raised by the non-measurability of sup x_t when t ranges through a continuum. A S.P. is "separable" when there exists (i) a (dense) denumerable subset S of the parameter set T and (ii) an $\mathfrak{A}$-set Z of P-measure zero such that, outside Z, and for every open interval I which meets T, the sets

$$\{x_t(\omega): t \in I \cap T\} \quad \text{and} \quad \{x_t(\omega): t \in I \cap S\}$$

have the same supremum and infimum. A stronger concept, "separability with regard to the closed sets", is also defined but is only rarely used later in the book.

If a S.P. $\{x_t: t \in T\}$ can be replaced by a separable one $\{\tilde{x}_t: t \in T\}$ on the same probability space and with the same parameter set, such that

$$\mathbf{P}(\{x_t(\omega) \neq \tilde{x}_t(\omega)\}) = 0, \quad t \in T,$$

and such that each $\tilde{x}_t$ is measurable with regard to the completion of $\mathfrak{B}(x_t: t \in T)$ under $\mathbf{P}$, then the original S.P. can be replaced by this separable "standard modification" for all practical purposes. That this is always possible is the content of Theorem 2.4 of Chapter II; as the author says, "this result is the best that could be hoped for". In constructing such a separable "standard modification" of a S.P. it will in general be necessary to allow the new random variables to assume infinite values, even if the original random variables were finite-valued.

If it is desired to integrate the sample functions of a S.P. then separability is not enough. Accordingly, Doob calls a S.P. "measurable" when (i) T is Lebesgue measurable and (ii) $x_t(\omega)$ is measurable in (t, ω) with regard to $\mathfrak{L} \otimes \mathfrak{A}$ (presumably after the latter has been completed under the product of Lebesgue measure and P-measure) (here $\mathfrak{L}$ is the Borel field of Lebesgue measurable subsets of R). In the important Theorem 2.6 the author shows that under a condition (**) there is a separable measurable "standard modification" of a S.P. $\{x_t: t \in T\}$ whenever T is Lebesgue measurable. (In general the new S.P. will not be finite-valued.) A sufficient condition (**) is that x_t be continuous-in-probability at almost all $t \in T$. If $T \in \mathfrak{L}$ and is of positive Lebesgue measure, it is shown by an example that some such condition as (**) is essential; thus, unlike separability, measurability imposes a restriction on the finite-dimensional distributions (though one which will usually be satisfied in practice).

There follow some theorems on the integrals of sample functions, and a summary without proofs of the earlier solution to the above problems by Doob and Ambrose. (This second method, by extension of the basic probability space, is not used in the book.) The remainder of Chapter II is devoted to the definitions and simplest properties of the eight classes of S.P.'s discussed in the rest of the book. The reader will find his comprehension of the theory so far developed severely and excitingly tested when he works through Doob's proof that a Markov process when reversed in time is still a Markov process.

An adequate review of the remaining ten chapters (500 pages) would be impossible within present space- and time-limitations, and only an indication of their contents can be given. Chapter III ("Processes with mutually independent random variables") deals in 50 pages very throughly with the zero-one law, series of independent random variables and their convergence, the law of large numbers, infinitely divisible distributions (an improved treatment utilising Doob's inequalities for characteristic functions), the central limit theorem and a theorem on the futility of gambling systems. "Stable laws" and the law of the iterated logarithm are about the only topics not discussed in this field.

Chapter IV ("Processes with mutually uncorrelated or orthogonal random variables") is a probabilistic version of the theory of orthogonal functions. For the proof of the Riesz-Fischer theorem and an important theorem on power-series required later for the factoring of a spectral density the reader is referred to classical texts on orthogonal functions and trigonometrical series.

Chapter V ("Markov processes—discrete parameter") is concerned with stationary Markov processes for which the phase-space X (in which the random variables assume their values) is abstract and T is $\{n: n=1, 2, 3, \cdots\}$. Strictly speaking the author here departs from his definition of a Markov process in Chapter II (in which X was the real line) but there is never any doubt about the existence of the conditional probabilities, etc., for these are supposed given a priori and a probability-measure is built up from them as in Tulcea's theorem. Specifically it is supposed that we are given a function p on $X \times \mathfrak{F}_X$ ($\mathfrak{F}_X$ being a Borel field of subsets of X) which determines for each $\xi \in X$ a probability-measure on $(X, \mathfrak{F}_X)$ and for each $A \in \mathfrak{F}_X$ a function measurable with regard to $\mathfrak{F}_X$. $p^n(\xi, A)$ is the corresponding n-step transition probability. The author commences with the classical case in which $X = \{1, 2, \cdots, N\}$. After some interesting remarks on card-shuffling he then proceeds straight to the general case, giving a treatment based on but more general than that of W. Doeblin [Bull. Math. Soc. Roumaine Sci. 39, no. 1, 57–115, no. 2, 3–61 (1937)], and depending on the following "Hypothesis D": There is a non-null finite-valued measure φ on $(X, \mathfrak{F}_X)$, an integer $\nu \geqq 1$ and a positive ϵ such that $p^\nu(\xi, A) \leqq 1-\epsilon$ if $\varphi(A) \leqq \epsilon$.

This is a limitation even in the important special case when X is countable, and as no separate discussion is given of the latter, the beginner (who will, however, not read so far as this) might conclude that Hypothesis D is necessary there. But this hardly matters, as an excellent discussion of the countable case is already available in another book [W. Feller, Probability theory and its applications, v. I, Wiley, New York, 1950; these Rev. 12, 424]. Doeblin's treatment of the general case is further amplified by an analysis of the possible classes of D-triples (φ, ν, ϵ) associated with such a process, and the chapter concludes with very general forms of the law of large numbers and the central limit theorem for Markov processes.

In Chapter VI ("Markov processes—continuous parameter") T is the half-line $\{t: t \geqq 0\}$, the processes are (to begin with) of the "jumping" type, and once again the chapter begins with a careful account of the properties of such systems when X is finite, and the emphasis is again on the stationary case. After dealing with a finite phase-space the author then allows X to be any Borel set in k-dimensional Euclidean space. Hypothesis D again appears, but only at one point not affecting the main argument. The author claims that the results of this section are valid "with moderate changes even if X is an abstract space (with no topology defined on it)" and the restriction to a Euclidean X is presumably to make available probabilistic methods. (As

noted above, Tulcea's theorem will not suffice for the construction of a probability measure if T is a half-line and X is abstract and if only "forwards" transition-functions are given. When X is Euclidean, the required measure can be constructed by an application of Kolmogorov's theorem.) The author discusses the character of the discontinuities of the sample functions and extends his (1945) method of transforming a "dishonest" into an "honest" Markov process. Some topics receiving much attention at the moment (semi-group methods, Lévy's analysis of the possible pathologies, and Feller's necessary and sufficient condition for the occurrence of an infinity of transitions in a finite time) are mentioned only in the Appendix. Just after the book went to press, the author's theorems on the limiting behaviour of the transition-functions near $t=0$ were sharpened by A. N. Kolmogorov [Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 53–59 (1951); these Rev. **14**, 295]. The chapter concludes with an extension of important work by K. Itô [e.g., Mem. Amer. Math. Soc. no. **4** (1951); these Rev. **12**, 724] on Markov processes of the diffusion type and associated stochastic differential equations.

The hundred pages on "Martingales" (Chapter VII) form the core of the book. A S.P. $\{x_t: t \in T\}$ of integrable real-valued random variables is a (semi-)martingale when $\mathbf{E}(x_{t_{n+1}} | x_{t_1}, \cdots, x_{t_n}) = (\geqq) x_{t_n}$ (with probability one) whenever $t_1 < t_2 < \cdots < t_{n+1}$. When the parameter-set T is $\{m: m=1, 2, \cdots\}$, a (semi-)martingale can be viewed as a rationalisation of one's intuitive notion of (an advantageous) a fair game (x_m being the gambler's fortune after the mth play). This latter notion motivates the analysis throughout the chapter and leads to theorems of astonishing scope. Thus, (semi-)martingales should preserve their nature under transformations suggestively called "optional stopping" and "optional sampling", and this is proved (under suitable conditions). There follow fundamental inequalities for martingales and semi-martingales and some very general consequences in the form of convergence theorems for both types of S.P. The Appendix contains a detailed account of the links between these results and earlier work (particularly that of E. S. Andersen, B. Jessen, P. Lévy and J. Ville). There are applications to the theory of sums of independent random variables, the law of large numbers, integration theory, the theory of derivatives, the likelihood ratio in statistics, sequential analysis and (after a long discussion of continuous-parameter martingales, in which separability naturally plays a role) to sample-function continuity.

Chapter VIII ("Processes with independent increments") is the continuous-parameter equivalent of chapter III and contains an account of the Brownian-motion process and the Poisson process (with an interesting invariance theorem for the latter) before proceeding to detailed proofs of theorems due to Lévy concerning the "centering" of the general process with independent increments, the character of the distribution functions and the continuity of the sample functions.

Chapter IX ("Processes with orthogonal increments") is what the author would call the "wide sense" version of Chapter VIII. Throughout this and the remaining chapters of the book a consistent and elegant use is made of the fact that the random variables of finite (absolute) second moment on an arbitrary fixed basic probability space can be grouped into equivalence classes which constitute the elements of a not necessarily separable Hilbert space. Stochastic integrals of the Wiener type ($\int\Phi(t)dy_t$) with regard to a S.P. $\{y_t: t \in T\}$ of orthogonal increments are defined in the spirit of this idea, and a generalisation is then given of an extension by K. Itô [Proc. Imp. Acad. Tokyo **20**, 519–524 (1944); these Rev. **7**, 313] to integrals of the type $\int\Phi(t, \omega)dy_t$. In Doob's integral $\{y_t: t \in T\}$ is a martingale and for each t the random variable $\Phi(t, \omega)$ is (very crudely speaking) determined by the y's carrying a parameter-suffix $\leqq t$. The construction is such that Doob is then able to define $\int_a^t \Phi(s, \omega)dy_s$ so as to make this "indefinite integral" itself a martingale. Finally the author characterises those martingales which can be represented in this way, using a y-process of Brownian-motion type.

Chapters X and XI ("Stationary processes—discrete/continuous parameter") contain parallel results in the relation indicated. The discussion commences with a valuable account of 1-1 measure-preserving point transformations, measure-preserving set transformations, invariant sets and metric transitivity, and the equivalence of these concepts with their images in the theory of strictly stationary S.P.'s. There is also an account of isometric and unitary transformations in relation to "wide sense" stationary processes (S.P.'s stationary "to the second order"). The Birkhoff ergodic theorem is proved as a strong law of large numbers for strictly stationary processes, and the L^2-ergodic theorem as a LLN for stationary (wide sense) processes. The autocovariance function of a stationary (wide sense) process is represented as a Fourier transform and the Cramér-Loève harmonic analysis of a (wide sense) stationary process is obtained following a method of H. Cramér. All the analogous theorems for a continuous parameter are proved in Chapter XI, and in the Appendix the connexion with the von Neumann-Wintner and Stone representation theorems for unitary operators in Hilbert space is explained in detail. Each chapter contains a section on the estimation of the covariance and the spectrum (although of course the very recent work of U. Grenander and M. Rosenblatt [reviewed fifth below] is not covered) and an analysis of linear operations and rational spectral densities. Chapter XI concludes with an account of the representation theorem for S.P.'s with stationary (wide sense) increments.

The final Chapter (XII) is concerned with "Linear least squares prediction—Stationary (wide sense) processes". This rather specialised topic is, as the author remarks, somewhat out of place in the book but it follows very naturally on the immediately preceding chapters. General solutions of the prediction problem are given in both the discrete- and continuous-parameter case, and the chapter concludes with the approximation of an arbitrary random variable (with finite second absolute moment) by linear combinations of the random variables constituting a stationary (wide sense) S.P., and with a short account of multidimensional prediction.

There is a substantial historical Appendix and a bibliography of 160 items. The index is largely one of definitions, and this review will (it is hoped) be found more informative than the list of contents.

Very few readers will work steadily through the whole of this difficult book, but it cannot fail to exercise a decisive influence on the development of its subject.

D. G. Kendall (Oxford).

Doob, J. L. The measure-theoretic setting of probability theory. Ann. Soc. Polon. Math. **25** (1952), 199–209 (1953).

En calcul des probabilités, de nombreux développements, en particulier sur les processus stochastiques et les prob-

abilités conditionnelles, ne sont rigoureux que sous certaines hypothèses concernant les probabilités ou mesures intervenant, hypothèses que l'on omet en général de formules, et qu'en fait n'ont pas toujours été dégagées d'une façon satisfaisante, malgré les efforts de Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933] et de Gnedenko et Kolmogoroff [Limiting distributions of sums of independent random variables, Gostehizdat, Moscow-Leningrad, 1949; ces Rev. **12**, 839]. L'auteur le rappelle et signale ses propres efforts, qu'il a developpés dans un ouvrage en préparation [aujourd'hui paru: voir le livre analysé ci-dessus], et dont il donne un exposé sommaire qu'il serait difficile de résumer ici, mais qui porte en particulier sur les points suivants: triplets parfaits de Gnedenko-Kolmogoroff (un triplet étant le système d'un ensemble Ω, d'un corps de Borel G de sous-ensembles de Ω, et d'une mesure de probabilité Pr définie sur G), processus stochastiques dans un espace fonctionnel (cas où Ω est un espace fonctionnel), espérances mathématiques et lois de probabilité conditionnelles, mesurabilité, continuité des processus stochastiques, etc. *R. Fortet* (Paris).

✓ ***Itô, Kiyoshi. Kakuritsu-ron. [Theory of probability.]** Iwanami Shôten, Tokyo, 1953. 6+405 pp. Yen 650.

This book gives a systematic exposition of stochastic processes based rigorously upon Kolmogoroff's axioms of probability. Although it is written independently of the recent book by J. L. Doob reviewed second above, there is a good deal of similarity, in the style and in the topics expounded, between these two books. Thus we see that stochastic processes have reached the epoch of development when a unified treatment of them is possible.

The contents of the book read as follows. Chap.1. Fundamental concepts. Chap. 2. Probability vectors and their distributions. Chap. 3. The sum of independent random variables. Chap. 4. Additive processes. Chap. 5. Wiener processes. Chap. 6. Stationary processes. Chap. 7. Markoff processes. Appendices on measure theory.

Chap. 1–3 give prerequisites from probability theory culminating in the law of large numbers and the central limit theorem. A stochastic process is a random variable $X(t)=X(t, \omega)$ depending upon a time parameter t (here ω denotes probability parameter). It is called additive (Chap. 4) if, for $t_1<t_2<\cdots<t_n$, the random variables $(X(t_i)-X(t_{i-1}))$ are mutually independent. It is a normal process (Chap. 4) if its sample process $X(t, \omega_0)$ is continuous in t almost certainly (i.e. for almost all ω_0). An additive process $X(t)$ is called a Lévy process (Chap. 4) if $\lim_{t\to t_0} \Pr(|X(t)-X(t_0)|>\epsilon)=0$ $(\epsilon>0)$ and if its sample process is discontinuous in t at most of the first kind almost certainly. A Lévy process is called (Chap. 4) a Poisson process if its sample process is a right-continuous step function increasing with a definite positive saltus almost certainly. The resolution, into the Poisson processes and the normal processes, of the Lévy process and its converse is expounded rigorously, refining the author's dissertation [Jap. J. Math. **18**, 261–301 (1942); these Rev. **7**, 312]. The Wiener process (=the Brownian motion) is a normal process which is stationary in time (Chap. 5). It is discussed here by making use of the two notions, "the system of normal random variables" and "the Wiener integral". The latter notion is extended (Chap. 7) to a more general "stochastic integral" [cf. Itô, Mem. Amer. Math. Soc., no. 4 (1951); these Rev. **12**, 724] which enables one to integrate "the stochastic differential equations". The s.d.e. is introduced by the author to define a fairly general class of Markoff processes, viz. stochastic processes whose futures are determined from the knowledge of the present states irrespective of their past histories. Chap. 6 is devoted to the exposition of results due to G. D. Birkhoff, Khintchine, Lévy, Doob, Loève, Maruyama, etc., together with Kolmogoroff's extrapolation and interpolation of stationary processes. *K. Yosida* (Osaka).

Maruyama, Gisirô. Markov processes and stochastic equations. Nat. Sci. Rep. Ochanomizu Univ. **4**, 40–43 (1953).

Announcement without proofs of several results on one-dimensional Markov processes of diffusion type. Given a pair of diffusion coefficients, Itô has constructed a corresponding diffusion process [Proc. Japan Acad. **22**, nos. 1–4, 32–35 (1946); these Rev. **12**, 191]. The author states that this process can also be obtained by solving a specified stochastic difference equation. Under stated conditions, a sequence of diffusion processes approximating a given diffusion process determines distributions approximating those of the latter process. Conditions are stated under which a Markov process can be identified with a diffusion process obtained by Itô's method. Such conditions have also been obtained by the reviewer in the book reviewed third above. *J. L. Doob* (Urbana, Ill.).

Pennanéac'h, F. Processus de Poisson. Cahiers Rhodaniens **4**, 1–11 (1952).

The Poisson process for continuous time is derived by postulating the order of magnitude of the probability of one event and of more than one event in a small time interval. The treatment is essentially the same as in Feller, "An introduction to probability theory . . ." [v. 1, Wiley, New York, 1950, Chapter 17; these Rev. **12**, 424]. The author also shows that the conditional distribution of events in an interval of time given that N events have occurred is that of N points chosen independently with uniform distribution in the interval. This point is also discussed in Chapter 8 of Doob, Stochastic processes [reviewed fourth above]. *J. L. Snell* (Princeton, N. J.).

Grenander, Ulf, and Rosenblatt, Murray. Statistical spectral analysis of time series arising from stationary stochastic processes. Ann. Math. Statistics **24**, 537–558 (1953).

The authors aim to cover those cases in which it is not appropriate to test the fit of a stationary time series model by considering a finite number of parameters, the model being too complex. The method proposed is essentially an application of the Kolmogorov test (of the greatest deviation of an empirical distribution function from expectation) to the integrated periodogram. This application is carefully justified for both Gaussian and non-Gaussian processes, for the case in which the observed series consists of the deviations from a regression on a finite number of given sequences, and for various modifications of the spectral estimate. All results are necessarily asymptotic. *P. Whittle.*

Grenander, Ulf, and Rosenblatt, Murray. An extension of a theorem of G. Szegö and its application to the study of stochastic processes. Trans. Amer. Math. Soc. **76**, 112–126 (1954).

The authors investigate the minimum of the quadratic form $c'Mc$ (for varying c) when M is a matrix with (p, q)th element given by $\int_{-\pi}^{\pi} e^{i(p-q)\lambda} f(\lambda)d\lambda$, $p, q=0, \cdots, n$, and f is

a specified non-negative integrable function. Constraints of the form $P^{(k)}(a_j)=b_j^k$ are imposed, where $|a_j|\leqq 1$ and $P(w)=\sum_0^n c_j w^j$. Szegö [Math. Z. 6, 167–202 (1920); 9, 167–190 (1921)] treated the special case in which there is only one a_j and the condition is $P(a_j)=1$. The authors find expressions for the minimum and for its limit when $n\to\infty$. The rate of approach to the limit is evaluated in some cases. Applications are made to prediction and estimation theory of stochastic processes. *J. L. Doob* (Urbana, Ill.).

Moy, Shu-Teh Chen. Measure extensions and the martingale convergence theorem. Proc. Amer. Math. Soc. 4, 902–907 (1953).

Andersen and Jessen [Danske Vid. Selsk. Math.-Fys. Medd. 22, no. 14 (1946); these Rev. 7, 421] proved the following theorem. Theorem A. Let $F_1\subset F_2\subset\cdots\subset F_n\subset\cdots$ be a nondecreasing sequence of Borel fields of subsets of a nonempty set Ω. Let P be a probability measure defined on the smallest Borel field F_∞ containing all the F_n's. Let ϕ be a bounded, countable additive set function defined on F_∞. Let P_n, ϕ_n be the contractions of P, ϕ to F_n respectively, and suppose that each ϕ_n is absolutely continuous with respect to P_n. Let x_n be the derivative of ϕ_n relative to P_n. Then $\{x_n\}$ converges, except on a set of P-measure 0 to the derivative of the P-continuous part of ϕ relative to P. That the x_n's converge follows from a result of Doob's [Trans. Amer. Math. Soc. 47, 455–486 (1940); these Rev. 1, 343]: Theorem B. If $\{x_n: n\geqq 1\}$ is a stochastic process such that $E\{x_n|x_1,\cdots,x_{n-1}\}=x_{n-1}$ and $\sup_{n\geqq 1}E\{|x_n|\}<\infty$, then x_n converges with probability one. Theorem B cannot be obtained directly from theorem A for general probability spaces but it is shown in this paper that if the x_n's in theorem B are coordinate variables in the space of all real sequences $\{\xi_n\}$ then the convergence follows from theorem A. The general case can then be proved by making a suitable mapping of the probability space onto a coordinate space and applying this result. *J. L. Snell* (Princeton, N. J.).

Kolmogoroff, A. A. Stationary sequences in Hilbert spaces. Trabajos Estadistica 4, 55–73, 243–270 (1953). (Spanish)

Translated from Byull. Moskov. Gos. Univ. Matematika 2 (1941); these Rev. 5, 101.

Tortrat, A. Les processus strictement stationnaires de Markoff et leurs corrélations. J. Math. Pures Appl. (9) 32, 281–333 (1953).

The author discusses stochastic processes of the form $\{f[x(t)], 0\leqq t<\infty\}$, where the $f[x(t)]$ process is a strictly stationary Markov process, with particular emphasis on the correlation functions. (In spite of the title of the paper, the $f[x(t)]$ process need not be a Markov process.) The principal tools are the semigroups generated by the $x(t)$ processes in the usual way. The spectral distribution function of an $f[x(t)]$ process is expressed in terms of the resolvent of the infinitesimal generator of one of these semigroups. There are many illuminating examples. *J. L. Doob.*

Kampé de Fériet, Joseph. Fonctions aléatoires harmoniques dans un demi-plan. C. R. Acad. Sci. Paris 237, 1632–1634 (1953).

If the boundary function in the Poisson integral formula for the upper half plane is a random function, the value $u(x, y)$ of the corresponding harmonic function will be a random variable. The covariance function of the $u(x, y)$ stochastic process is computed under the hypothesis that the assigned boundary function, considered as a stochastic process, is stationary of order 2. *J. L. Doob.*

Sirao, Tunekiti. On some asymptotic properties concerning homogeneous differential processes. Nagoya Math. J. 6, 95–107 (1953).

Let $\{x(t), 0\leqq t<\infty\}$ be a stochastic process with stationary independent increments, right continuous sample functions, and with $E\{x(t)\}=0$, $E\{x(t)^2\}=t$. The author proves (under various supplementary hypotheses on the distribution of $x(1)$) that $x(t)\leqq t^{1/2}\phi(t)$ for sufficiently large t, with probability 1, where ϕ is right continuous and monotone nondecreasing, if and only if

$$\int^\infty \frac{1}{t}\varphi(t)e^{-\frac{1}{2}\varphi(t)^2}dt<\infty.$$

The result is familiar in the Brownian motion case. *J. L. Doob* (Urbana, Ill.).

Darling, D. A., and Siegert, A. J. F. The first passage problem for a continuous Markov process. Ann. Math. Statistics 24, 624–639 (1953).

Let $X(t)$ be a strongly continuous temporally homogeneous Markov process. Define

$$T=T_{ab}(x)=\sup\{t|a>X(\tau)>b, 0\leqq\tau\leqq t\},$$
$$P(x|y,t)=\Pr\{X(t+s)<y|X(s)=x\},$$
$$F_{ab}(x|t)=\Pr\{T<t\}.$$

Let P and F have densities

$$p(x|y,t)=\partial P/\partial y,\quad f_{ab}(x|t)=\partial F/\partial t.$$

Laplace transforms are denoted by (^); thus

$$\hat{p}(x|y,\lambda)=\int_0^\infty e^{-\lambda t}p(x|y,t)dt.$$

Also $f_{ab}(x|t)=f^+{}_{ab}(x|t)+f^-{}_{ab}(x|t)$, where f^+ and f^- are right and left absorption densities, respectively. The treatment makes use of the fact that $\hat{p}$ is a product, $\hat{p}=u(x)u_1(y)$, $y>x$; $\hat{p}=v(x)v_1(y)$, $y<x$. Then

$$\hat{f}^+=[v(b)u(x)-u(b)v(x)]/[u(a)v(b)-u(b)v(a)],$$
$$\hat{f}^-=[u(a)v(x)-v(a)u(x)]/[u(a)v(b)-u(b)v(a)],$$

$\hat{f}=\hat{f}^++\hat{f}^-$. In most cases of interest, p satisfies

$$\partial p/\partial t=A(x)\partial p/\partial x+\tfrac{1}{2}B^2(x)\partial^2 p/\partial x^2,$$
$$p(-\infty)=p(\infty)=0,\quad p(x|y,0)=\delta(x-y)$$

(Dirac δ). Then u and v above can be taken as any two linearly independent solutions of $Adw/dx+\frac{1}{2}B^2d^2w/dx^2-\lambda w=0$. Applications are made to the Einstein-Wiener process, the Uhlenbeck process, and sequential analysis. Letting x_1, $x_2,\cdots$ be independent random variables obeying the central limit theorem, $Ex_i=0$, $Ex_i^2=\sigma^2$, the asymptotic distribution of max $\{|S_n|n^{-\frac{1}{2}}, n_1\leqq n\leqq n_2\}$ is found by reducing the problem to a consideration of the Uhlenbeck process. Moments of T and the distribution of the range of $X(t)$ are studied. *T. E. Harris* (Santa Monica, Calif.).

Bush, Robert R., and Mosteller, Frederick. A stochastic model with applications to learning. Ann. Math Statistics 24, 559–585 (1953).

Although in §§1–3 the authors discuss a more general problem, the one dealt with in the main body of this paper is essentially as follows. The subject of a "learning" experiment is required to make a succession of choices between alternatives A_1 and A_2. Let p_{n-1} be the probability that he will choose A_1 at the nth trial (so that $1-p_{n-1}$ is the prob-

ability of his choosing A_2). If A_1 is chosen, then the subject ("learning from experience") is supposed to choose A_1 at the $(n+1)$st trial with probability $p_n=a_1+\alpha_1 p_{n-1}$, while if A_2 is chosen at the nth trial then $p_n=a_2+\alpha_2 p_{n-1}$. Thus, if p_0 is given, the sequence of random variables $p_1, p_2, p_3, \cdots$ constitutes a Markov process with a continuum of possible states, and this is the model adopted by the authors to represent certain simple "learning" situations. They state that T. E. Harris [same Ann. **23**, 141 (1952)] has proved that the distribution of p_n will converge to a limit-distribution which is independent of p_0 if $0<\alpha_i<1$ and $0<a_i<1-\alpha_i$ $(i=1, 2)$. In §4 the authors give upper and lower bounds for $E(p_n)$ as a function of n, together with some "Monte Carlo" data.

They then turn to the problem of estimating the parameters p_0, a_i, α_i $(i=1, 2)$ from observed sequences of choices. A number of special cases of the estimation problem are discussed in detail, especially (i) $a_i=1-\alpha_i$ $(i=1, 2)$ and (ii) $\alpha_1=\alpha_2$. The resulting formulae are applied to various sets of "learning" data: dogs learning to avoid electric shocks, rats in a T-maze learning to find food, and students learning to make the choice which will lead to a small monetary reward. *D. G. Kendall* (Oxford).

Karlin, Samuel. Some random walks arising in learning models. I. Pacific J. Math. **3**, 725–756 (1953).

[For the psychological problem which gave rise to the present investigation see the paper reviewed above.] The author is concerned with the Markov process $\{x_n: n\geqq 1\}$ having the following transition-probabilities:

$$\mathbf{P}(x_{n+1}=\alpha+(1-\alpha)x_n|x_n)=\phi(x_n),$$
$$\mathbf{P}(x_{n+1}=\sigma x_n|x_n)=1-\phi(x_n).$$

Here α and σ are constants in the open interval (0, 1) and $0\leqq x\leqq 1$, $0\leqq\phi(x)\leqq 1$. [The reader should attend to a variation in notation from α to $1-\alpha$ throughout the paper.] Several different assumptions are made about the form of the function ϕ, of which the chief ones are: (1) $\phi(x)\equiv x$; (2) $|\phi(x)-\phi(y)|\leqq\mu<1$; (3) $\phi(x)\equiv 1-x$; (4) ϕ is linear; (5) $0<\delta\leqq\phi(x)\leqq 1-\delta<1$. For brevity only case (1) will be summarised here. The elementary transitions of the process determine a linear bounded transformation T from the space of measures (on the unit interval) into itself, and T is adjoint to the bounded linear transformation U,

$$(U\pi)(t)=(1-t)\pi(\sigma t)+t\pi(\alpha+(1-\alpha)t),$$

which maps $C[0, 1]$ into itself. It is shown that U preserves (i) $\pi(0)$ and $\pi(1)$, (ii) positivity, (iii) positive monotony and (iv) positive monotone convexity. With the aid of these results, a bound for $(U^n\pi)'(1)$ and a theorem of R. Bellman asserting that there is at most one continuous solution of $U\pi=\pi$ such that $\pi(0)=0$ and $\pi(1)=1$, the author shows that when $\pi(t)\equiv t^r$ then $U^n\pi$ (as $n\to\infty$) converges in the topology of $C[0, 1]$ to a limit-function θ which is the same for every $r\geqq 1$. It follows that U^n converges strongly, that $(U^n\pi, F)\equiv\int U^n\pi dF$ converges for every distribution F and that T^nF enjoys a weak* convergence equivalent to convergence in the sense of distributions. The limiting distribution is

$$(**)\qquad \Delta_1(x)\int\theta dF+\Delta_0(x)\int(1-\theta)dF,$$

where $\Delta_y(x)$ is the distribution concentrated at $x=y$. Thus $x=0$ and $x=1$ act as absorbing barriers.

[The process is a bounded martingale when $\alpha+\sigma=1$ and a bounded upper or lower semi-martingale otherwise. Known theorems would thus seem to imply the convergence of the random variable x_n with probability one, and (**) will be the distribution-function of the limiting random variable.] The author gives an analogous discussion of cases (2) to (5) [and it may be noted that martingale theory would here be of no assistance]; in case (3), for example, $x=0$ and $x=1$ act as reflecting barriers. In case (5), T^nF converges in distribution to a limit $F_{\sigma,\alpha}$ which is independent of F, and the author finds the limiting distribution to be singular (and spread on a Cantor-like set) when $\sigma<\alpha$ and to be either absolutely continuous or singular when $\sigma\geqq\alpha$ (he conjectures that the first alternative is always correct). *D. G. Kendall* (Oxford).

Muroga, Saburo. On the capacity of a discrete channel. I. Mathematical expression of capacity of a channel which is disturbed by noise in its every one symbol and expressible in one state diagram. J. Phys. Soc. Japan **8**, 484–494 (1953).

The author shows how to calculate the capacity of a noisy discrete channel, avoiding certain difficulties in the method suggested by Shannon [Bell System Tech. J. **27**, 379–423 (1948); these Rev. **10**, 133]. *J. L. Doob.*

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***Woodward, P. M. Probability and information theory, with applications to radar.** McGraw-Hill Book Co., Inc., New York; Pergamon Press Ltd., London, 1953. x+128 pp. $4.50.

1) An introduction to probability theory. 2) Waveform analysis and theory. 3) Information theory. 4) The statistical problem of reception. 5) Simple theory of radar reception. 6) The mathematical analysis of radar information. 7) The transmitted radar signal. *Table of contents.*

Mathematical Statistics

Rosenbaum, S. Tables for a nonparametric test of dispersion. Ann. Math. Statistics **24**, 663–668 (1953).

As a test for the hypothesis H_0 that two populations π_1 and π_2, with continuous c.d.f.'s are equally distributed, against the alternative that π_2 is more scattered than π_1, the author suggests the number of values in a random sample of size m from π_2 that lie outside those of a random sample of size n from π_1. Using the fact that the distribution of the test when H_0 is true is identical with one obtained by Wilks in connection with tolerance limits [same Ann. **13**, 400–409 (1942); these Rev. **4**, 165] the author gives critical 5% and 1% values for $m=2(1)50$ and $n=2(1)50$. *D. M. Sandelius* (Göteborg).

Savage, I. Richard. Bibliography of nonparametric statistics and related topics. J. Amer. Statist. Assoc. **48**, 844–906 (1953).

The bibliography contains a listing by author and a subject index.

Williams, E. J. A method of analysis for double classifications. Australian J. Appl. Sci. **4**, 357–370 (1953).

An approximate analysis is given for double classification data (p groups and q treatments) with unequal numbers in the subclasses. Given n_{ij} observations for the ith group and jth treatment with $x_{i.}$ the over-all mean for group i and $x_{.j}$

the over-all mean for treatment j. The approximate effect for treatment j (adjusted for groups) is $t_j' = \sum_i n_{ij}(x_{ij} - x_{i.})/n_{.j}$, where $n_{.j} = \sum_i n_{ij}$. An approximate adjusted sum of squares attributable to treatments, adjusted for groups, is

$$T' = \sum n_{.j} t_j'^2.$$

Here t_j' and T' can be further adjusted if the n_{ij} are decidedly unequal:

$$t_j'(\text{adj}) = t_j'/(1 - \tfrac{1}{2}d); \quad T'(\text{adj}) = (q-1)T'/(q-d-1),$$

where $d+1 = \sum_{i,j}(n^2_{ij}/n_{i.}n_{.j})$. A procedure is also given to make individual treatment comparisons.

R. L. Anderson (Raleigh, N. C.).

Ogasawara, Tôzirô, and Takahashi, Masayuki. Orthogonality relation in the analysis of variance. I. J. Sci. Hiroshima Univ. Ser. A. **16**, 457–470 (1953).

The data on which an analysis of variance is to be performed are a set of N values of a random variable x with an index set of subscripts $(1, 2, \cdots, N)$ which are divided into classes according to one or more criteria of classification. The sums of squares, S_A, between classes of any one kind, A, is a quadratic form with a projective matrix. This leads the authors to associate a projective operator P_A with S_A such that $\|P_A x\|^2 = S_A$. This paper studies the properties of such operators. The operators are defined by means of appropriate vectors in the N-dimensional Euclidean space of the x's and then follows a body of 19 theorems with numerous lemmas and corollaries that cannot be described in any detail here. The principal concern is the derivation of conditions on the classes according to two or more criteria of classification which result in orthogonality in the designs. Commutativity of operators corresponding to different criteria of classification has importance in this connection and this property is studied. The dependence of these operators is extended to interaction sums of squares and their properties with respect to orthogonality are investigated. There are applications to factorial designs.

C. C. Craig (Ann Arbor, Mich.).

Ogasawara, Tôzirô, and Takahashi, Masayuki. Orthogonality relation in the analysis of variance. II. J. Sci. Hiroshima Univ. Ser. A. **17**, 27–41 (1953).

This is a continuation of the paper reviewed above. The authors study the conditions that a classification C can be decomposed into a number of classifications $A_1, \cdots, A_p$ in the sense that $P_C = \sum_{i=1}^{p} P_{A_i}$. From the first paper $p \geqq 3$ and the product of the degrees of freedom for A_1 and A_2 does not exceed the product of the degrees of freedom for the remaining classifications; if equality holds then the interaction classification A_1A_2 is decomposed into $A_3, A_4, \cdots, A_p$. The structure of such decompositions for the degrees of freedom of $A_1 = p-2, p-3, p-4$ are studied and the results are applied for $p = 4, 5, 6$. Then the decompositions of interactions are investigated. There are applications to the construction of complete sets of orthogonal Latin squares.

C. C. Craig (Ann Arbor, Mich.).

Kamat, A. R. Incomplete and absolute moments of the multivariate normal distribution with some applications. Biometrika **40**, 20–34 (1953).

This paper deals with the evaluation of the separate contributions from each quadrant to the absolute moments of multivariate normal distributions.

The method adopted by the author is different from Nabeya's [Ann. Inst. Statist. Math., Tokyo **3**, 2–6 (1951); **4**, 15–30 (1952); these Rev. **13**, 570; **14**, 569], who treated merely the problems of evaluating absolute moments for the multivariate normal distributions. Concerning bivariate and trivariate cases the author gives concrete, useful formulae, and for the general multivariate cases illustrates the expansion in power series of correlation coefficients. Finally he considers the application of the results to various statistics related to absolute moments.

T. Kitagawa.

Kamat, A. R. On the mean successive difference and its ratio to the root mean square. Biometrika **40**, 116–127 (1953).

Let x_i $(i=1, 2, \cdots, n)$ denote a sequence of n normal variates with mean μ_i and a common variance σ^2. The author discusses the approximate distributions of the mean successive difference d, where $d = \sum_{i=1}^{n-1} |x_i - x_{i+1}|/(n-1)$, and of its ratio to the root mean square, d/s. The method is based upon the calculations of the first four moments and uses the results obtained by the author in the paper reviewed above.

T. Kitagawa (Fukuoka).

Dalenius, T. The multi-variate sampling problem. Skand. Aktuarietidskr. **36**, 92–102 (1953).

An expository discussion of various methods that have been proposed for determining sample sizes in the individual strata in stratified random sampling when the survey is intended to provide estimates for a number of different variates.

W. G. Cochran (Baltimore, Md.).

Bhattacharyya, A. On the uses of the t-distribution in multivariate analysis. Sankhyā **12**, 89–104 (1952).

The main result of this paper is summarized as follows. In a p-variate normal population with the population covariance matrix $((\sigma_{ij}))$ and a sample therefrom of size n with the sample covariance matrix $((s_{ij}))$, a parameter θ satisfying the equation (condition for the t-test)

$$\sum_{i,j=1}^{p} f_{ij}(\theta)\sigma^{ij} = 0, \quad ((\sigma^{ij})) = ((\sigma_{ij}))^{-1},$$

where the quantities $f_{ij}(\theta)$ satisfy the conditions of consistency: (1) $f_{ij}(\theta)^2 \geqq f_{ii}(\theta) f_{jj}(\theta)$, (2) rank $((f_{ij}(\theta))) = 2$, can be tested by the statistic written below and following the t-distribution (D.F. $= n-p$),

$$t = \frac{\left[\sum_{ij} f_{ij}(\theta) S_{ij}\right](n-p)^{\frac{1}{2}}}{2(|s_{ij}|)^{\frac{1}{2}}\left\{\sum_{j>i}\sum_{l>k}[f_{ik}(\theta)f_{jl}(\theta) - f_{il}(\theta)f_{jk}(\theta)]S_{ij \cdot kl}\right\}^{\frac{1}{2}}}.$$

Special cases of this result give us well-known results such as the Fisher t-test for regression coefficient, as well as new ones such as the generalisation of the Pitman-Morgan test.

T. Kitagawa (Fukuoka).

Rao, C. Radhakrishna. On transformations useful in the distribution problems of least squares. Sankhyā **12**, 339–346 (1953).

The author gives more elegant proofs, stated almost wholly in terms of matrices, for certain results given in his previous writings [Sankhyā **7**, 237–256 (1946); Advanced statistical methods in biometric research, Wiley, New York, 1952, §2d.1, pp. 58–65; these Rev. **8**, 41; **14**, 388]. The results concern the distributions of the conditional minima of certain quadratic forms in independent normal variates and are proved by means of suitable orthogonal transformations.

H. P. Mulholland (Birmingham).

Benard, A., and van Elteren, Ph. A generalization of the method of m rankings. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 358–369 (1953).

M. Friedman's statistic χ_r^2 for comparing m rankings [J. Amer. Statist. Assoc. **32**, 675–701 (1937)] is extended to the case of an $m \times n$ table in which the variate $X_{\mu\nu}$ in cell (μ, ν) is observed an arbitrary number $k_{\mu\nu} \geqq 0$ of times. The asymptotic distribution of the statistic in the case of equally probable rankings is obtained. *W. Hoeffding.*

Fisher, Walter D. On a pooling problem from the statistical decision viewpoint. Econometrica **21**, 567–585 (1953).

For certain cost and weight functions C and W, a limiting Bayes solution and an example of its application are given, for the problem of dividing k location-parameter populations into groups (the cost of any partition being reflected in C) within each of which it is desired (as reflected in W) that the spread of population means be small, on the basis of one observation (or sample mean) from each population.
J. Kiefer (Ithaca, N. Y.).

Cadwell, J. H. Approximating to the distributions of measures of dispersion by a power of χ^2. Biometrika **40**, 336–346 (1953).

The author derives formulas for transforming the distribution of a statistic u from a normal population such that u is approximately distributed like $(\chi^2/c)^a$ or cu^λ is approximately distributed as χ^2 with v degrees of freedom. The results are applied to the range, the mean range, the first quasi-range (the difference between the greatest-but-one and the least-but-one in an ordered sample), and the mean deviation usually for samples of size less than 20. In Table I are listed the values to 4 decimals of λ, and $\log c$, $n=2(1)20$ for the range, mean deviation, and $n=10(1)30$ for the first quasi-range, and for v ordinarily to 2 or 3 significant figures. In Table 2 are listed the values of λ, and $\log c$ to 4 decimals for the mean of m ranges, each for sample size n, where v is given to 2 or 3 significant figures, $m=2(1)5$, $n=2(1)10$. In Table 3 are listed the approximate upper 5% points of the ratio of the maximum value to the minimum value in a set of k independent ranges, for $k=2(1)12$, $n=3(1)12$, 12, 15, 20. In Table 4 the approximate upper 1% points for the range ratio are given for $k=2(1)12$, $n=3(1)10$. In Table 5 are given the approximate upper 5% points for the ratio of the maximum mean deviation to minimum mean deviation in a set of k independent values in a sample of n, $k=2(1)12$, $n=3(1)12$, 15, 20, 60, 60, ∞. Table 6 provides values of the approximate upper 1% points for the mean deviation ratio $k=2(1)12$, $n=3(1)10$. *L. A. Aroian.*

Cadwell, J. H. The distribution of quasi-ranges in samples from a normal population. Ann. Math. Statistics **24**, 603–613 (1953).

A method is presented for the evaluation of the probability density function of $w_r = x_{n-r} - x_{r+1}$, where $x_1, x_2, \cdots, x_n$ is an ordered random sample from a normal population. A table of percentage points and moment constants of w_1 is given for $n=10(1)30$. Comparing the variances of unbiased estimates of the population S.D. based on w_r with the variance of the "efficient" unbiased estimate $2^{-\frac{1}{2}}\{\Gamma(\frac{1}{2}(n-1))/\Gamma(\frac{1}{2}n)\}\{\sum(x_i-\bar{x})^2\}^{\frac{1}{2}}$, where $\bar{x}=n^{-1}\sum x_i$, the author finds w_0 most efficient for $2 \leqq n \leqq 17$, and w_1 most efficient for $18 \leqq n \leqq 31$. *D. M. Sandelius* (Göteborg).

Terpstra, T. J. The exact probability distribution of the T statistic for testing against trend and its normal approximation. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 433–437 (1 plate) (1953).

A recurrence relation for the distribution of the statistic T [see Terpstra, same Proc. **55** = Indagationes Math. **14**, 327–333 (1952); these Rev. **14**, 64] is derived for the case where all random variables are independent with a common continuous distribution function. Tables for certain small sample sizes are given. *W. Hoeffding.*

Vajani, Luigi. I criteri di R. A. Fisher per la scelta di una buona stima ed il metodo della massima verosimiglianza. Statistica, Bologna **13**, 311–342 (1953).

Kolmogoroff, A. N. Unbiased estimates. Amer. Math. Soc. Translation no. **98**, 28 pp. (1953).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 303–326 (1950); these Rev. **12**, 116.

Gihman, I. I. Some remarks on A. N. Kolmogorov's criterion of fit. Doklady Akad. Nauk SSSR (N.S.) **91**, 715–718 (1953). (Russian)

Suppose the line $-\infty < x < \infty$ divided into n disjoint intervals, and let $F_i(\theta)$ be the probability of the first i of these, where θ is a parameter to be estimated. Let $\bar{F}_i$ be the fraction of N independent observations from the above population which lie in the first i intervals, and let $\bar{\theta}$ be an estimator of θ. Under complicated conditions not easily stated here and even less easily ascertainable from the paper, the author states two theorems the conclusions of which are that the conditional probability of the simultaneous fulfillment of the inequalities $a_1 < N^{\frac{1}{2}}\{\bar{F}_i - F_i(\bar{\theta})\} < a_2$, $i=1, \cdots, n$, and $a_1 < 0 < a_2$, given that $N^{\frac{1}{2}}(\bar{\theta}-\theta) = z_N \to z$, converges in probability, as $N \to \infty$, to the value at the origin of a solution of the diffusion equation with suitable boundary conditions. *J. Wolfowitz* (Ithaca, N. Y.).

Wolfowitz, J. Estimation by the minimum distance method. Ann. Inst. Statist. Math., Tokyo **5**, 9–23 (1953).

The author demonstrates that the empirical cumulative distribution function of independent random variables converges with probability one to the mean theoretical cumulative distribution function. He uses this theorem to obtain consistent estimates of the parameters of a linear regression with both variables subject to error under essentially the same conditions as could be used with characteristic functions in a modification of the method of Neyman and Scott [Econometrica **16**, 1–32 (1948); these Rev. **9**, 600].
H. Rubin (Stanford, Calif.).

Chernoff, Herman. Locally optimal designs for estimating parameters. Ann. Math. Statistics **24**, 586–602 (1953).

Consider an experiment $\mathcal{E}$ the outcome of which is a random vector variable x whose distribution depends on parameters $\theta_1, \cdots, \theta_k$. The large-sample estimation power of this experiment is described by Fisher's information matrix $X = \|E\{(\partial L/\partial\theta_i)(\partial L/\partial\theta_j)\}\| = \|x_{ij}\|$ where L is the logarithm of the likelihood function. When several experiments $\mathcal{E}_i$ are randomized in proportions p_i, the matrix $\bar{X} = \sum p_i X_i$ reflects the power of the resulting mixed experiment, and also of a combined experiment where the $\mathcal{E}_i$ are repeated independently in proportions p_i. Let the problem be to estimate $\theta_1, \cdots, \theta_s$ $(s \leqq k)$ by unbiased estimates $t_1, \cdots, t_s$ in such a way that $\sum_1^s \operatorname{var} t_i$ be as small as possible. The design prob-

lem then consists in minimizing, with respect to $p_1, \cdots, p_k$ ($\sum p_i=1$), the sum $x^{11}+\cdots+x^{ss}$ of the s first diagonal elements in $\bar{X}^{-1}$. It is shown that for this purpose at most $k+(k-1)+\cdots+(k-s+1)$ positive p_i's are needed. When $\bar{X}$ is singular in the minimizing point, certain difficulties arise which are overcome by limit passages. The result is a generalization of one by Elfving [same Ann. **23**, 255–262 (1952); these Rev. **13**, 963], the most important new contribution being that each experiment may consist of several observations. *G. Elfving* (Helsingfors).

van der Waerden, B. L. Order tests for the two-sample problem. II. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 303–310 (1953).

The asymptotic distribution of Wilcoxon's test is investigated in the case of normal distributions with equal variances. In particular, the asymptotic relative efficiency of the Wilcoxon test with respect to the t-test is found to be $3/\pi$ independent of whether the two sample sizes are of equal magnitude or not. On the basis of investigations of various special cases two-sample tests are ranked according to their power as follows: (1) Student's t-test; (2) X-test [van der Waerden, same Proc. **55** = Indagationes Math. **14**, 453–458 (1952); **15**, 80; these Rev. **14**, 666]; (3) Wilcoxon's test; (4) Smirnoff's test [Bull. Math. Univ. Moscou **2**, no. 2 (1939); these Rev. **1**, 345]; (5) Wald-Wolfowitz run test [Ann. Math. Statistics **11**, 147–162 (1940); these Rev. **1**, 348]. *G. E. Noether* (Boston, Mass.).

van der Waerden, B. L. Order tests for the two-sample problem. III. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 311–316 (1953).

The investigations of the previous paper [see the preceding review] are continued for the case of non-normal distributions. In particular, the author gives an example where the power of rank order tests converges to 1 while the power of the t-test does not. The author draws a number of tentative conclusions about which test to use under given circumstances. *G. E. Noether* (Boston, Mass.).

Dixon, W. J. Power functions of the sign test and power efficiency for normal alternatives. Ann. Math. Statistics **24**, 467–473 (1953).

The power of the two-sided sign test [Dixon and Mood, J. Amer. Statist. Assoc. **41**, 557–566 (1946)] for the level of significance α is given by

$$\lambda(p)=\sum_{j=0}^{i}\binom{N}{j}[p^j(1-p)^{N-j}+p^{N-j}(1-p)^j]$$

where i is the largest integer such that

$$\sum_{j=0}^{i}\binom{N}{j}\left(\frac{1}{2}\right)^N\leq\tfrac{1}{2}\alpha$$

and N is considered fixed. The author tabulates $\lambda(p)$ for various sample sizes and α near .05 and .01 and compares $\lambda(p)$ with the power function of the t-test for samples from normal populations by means of a power efficiency function. The results would indicate decreasing power efficiency for increasing sample size, level of significance and alternatives. *R. P. Peterson* (Riverside, Calif.).

Sverdrup, Erling. Similarity, unbiassedness, minimaxibility and admissibility of statistical test procedures. Skand. Aktuarietidskr. **36**, 64–86 (1953).

The paper is mainly concerned with the theory of similar, unbiassed, most powerful, and uniformly most powerful tests in situations where exponential families of distributions play an important role. The functional completeness of the class of exponential functions, and an extension of the Neyman-Pearson Lemma are the main tools. The theory is applied to obtain conclusions, many of them new, about several double dichotomy problems, and about t- and F-tests. The terminal section of the paper is concerned with the minimax theory of tests under "simple" loss functions, without special reference to exponential families. *L. J. Savage* (Chicago, Ill.).

Box, G. E. P. Non-normality and tests on variances. Biometrika **40**, 318–335 (1953).

The author investigates the effect of non-normality on Bartlett's test, M_1, for the equality of variances in k groups of observations from non-normal parent populations. For large samples from non-normal populations it is shown that the distribution of M_1 depends strongly on the value of γ_2, the measure of kurtosis. In small samples experimental evidence points to the same conclusion. In fact M_1 is investigated as a test of normality. The author concludes that when little is known of the parent distribution a test of the homogeneity of variance in an analysis of variance should not be used since the usual test on the homogeneity of means is very little affected by non-normality of the parent population. *L. A. Aroian* (Culver City, Calif.).

David, H. A. The power function of some tests based on range. Biometrika **40**, 347–353 (1953).

The author appears to have compared the power of a test based upon the Studentized range (S-R) in the random model of the analysis of variance with that of the more conventional F-test. If the number of groups is "small," then, as would be expected, the loss in power in using the S-R is "small." It is asserted that the comparison is between the F-test and one based on a modified S-R in which the denominator is replaced by an average of the ranges (rendered unbiased). This involves the Chi-distribution as an approximation. A comparison between the power of the F-test and one based on the numerator of F but using the aforementioned modified denominator is given via an Incomplete Beta approximation. The systematic model is briefly treated and it is suggested that in certain cases the S-R test may have better power than the F-test. This is not incompatible with the known optimum properties of F. *H. Teicher* (Lafayette, Ind.).

Barton, D. E. On Neyman's smooth test of goodness of fit and its power with respect to a particular system of alternatives. Skand. Aktuarietidskr. **36**, 24–63 (1953).

The author considers the distribution of Neyman's smooth-test statistic ψ_k^2 for $k=1$ both under the null hypothesis and under some "natural" alternative hypotheses. He obtains the asymptotic power function and shows how to obtain the asymptotic power for certain common classes of alternative hypotheses. *H. Rubin* (Stanford, Calif.).

Mallows, C. L. Sequential discrimination. Sankhyā **12**, 321–338 (1953).

Let $X_1, X_2, \cdots$ be a sequence of normal random variables with known covariances and with means which under hypotheses H_j $(j=1, 2)$ are $\xi_{j1}, \xi_{j2}, \cdots$. The author considers the Wald likelihood ratio test for testing between H_1 and H_2 and shows that the logarithm of the likelihood ratio is a sum of independent (not necessarily identically distributed) random variables which may be treated by a

slight extension of Wald's methods to give properties of the test. [Remark: The author's generalization of Wald's equation is a simple consequence of martingale theory. The author's conclusion regarding the applicability in the present case of the limiting distribution of sample size in the case of (e.g.) identically distributed components is incorrect without further assumptions here limiting the relative sizes of components.] *J. Kiefer* (Ithaca, N. Y.).

Dumas, Maurice. Epreuve économique permettant de choisir entre deux hypothèses. C. R. Acad. Sci. Paris **237**, 1628–1629 (1953).

Suggests an extension of the sequential probability-ratio test to qualitative sampling inspection from a finite lot. *L. J. Savage* (Chicago, Ill.).

Dumas, Maurice. Epreuve économique permettant de choisir entre deux hypothèses. C. R. Acad. Sci. Paris **238**, 40–42 (1954).

Continues the note by the same author reviewed above. *L. J. Savage* (Chicago, Ill.).

Annis, M., Cheston, W., and Primakoff, H. On statistical estimation in physics. Rev. Modern Physics **25**, 818–830 (1953).

The authors attempt to give an exposition of the modern statistical methods of estimation. They try to develop estimation theory solely on the basis of the classical Bayes theorem approach. The failure of this approach led to the development of the modern theory of statistical inference. It is regrettable that the authors present an outmoded approach but create the impression that they give an account of the present state of the theory by references to the work of R. A. Fisher and H. Cramér. An up-to-date presentation of the theory of statistical estimation for physicists would be highly desirable. *E. Lukacs.*

Quenouille, M. H. Modifications to the variate-difference method. Biometrika **40**, 383–408 (1953).

The author considers a time series made up of a trend and a random component, employing quantities V_i which are weighted estimates of the variances of ith differences of observations. Given that trend is eliminated in the $(m+1)$th differences, an estimate is obtained from $V_{m+1}\cdots V_{m+p}$ of the variance of the random component, this estimate being approximately the extrapolated value of V_i at $i=0$. A sampling variance is attached to the estimate, and similar estimates are made of trend in the series V_i. These results are extended to the estimation of serial covariances, and the effect of serial correlation in the original series is briefly considered. Several numerical examples are included. *P. Whittle* (Wellington).

Blassel, Pierre. Erreur due à une durée d'intégration finie dans la détermination des fonctions d'autocorrélation. Ann. Télécommun. **8**, 406–414 (1953).

Mathematical Economics

Karlin, Samuel. Continuous games. Proc. Nat. Acad. Sci. U. S. A. **37**, 220–223; errata, 381 (1951).

The author indicates how to derive results, both qualitative and quantitative, on continuous zero-run two-person games by using the theory of cones in Banach spaces. His method involves changing the space of strategies and studying the effect of this change on the game. The results are stated without proof and do not lend themselves to condensed reproduction.

Several applications are mentioned; we quote the following: Let the game be on the unit square and let the kernel $M(x, y)$ be such that $d^nM(x, y)/dy^n$ is non-negative and continuous in both variables; then there exists an optimal strategy for player 1 (the maximizing player) with at most n points of increase. *A. Dvoretzky* (New York, N. Y.).

***Karlin, Samuel. On a class of games.** Contributions to the theory of games, vol. 2, pp. 159–171. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Let the kernel $K(x, y)$ of a zero-sum two-person game on the unit square have continuous nth order derivatives and let $d^nK(x, y)/dy^n \geq 0$. For $n \leq 4$ the author proves that player 1 (the maximizing player) has an optimal strategy consisting of at most n points of increase. The proof depends on an enumeration of cases and the author states that the result can be extended to general n. [In the paper reviewed above the author stated the above result for general n even under somewhat weaker conditions; no proof was given and the method indicated was quite different from the present one.] It is also proved (this part is easy) that player 2 (the minimizing player) has an optimal strategy consisting of at most $n/2$ points of increase, provided we make the convention that the end-points 0 and 1 are each counted as half a point. *A. Dvoretzky* (New York, N. Y.).

HAVE ***Karlin, Samuel. Reduction of certain classes of games to integral equations.** Contributions to the theory of games, vol. 2, pp. 125–158. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The paper deals with zero-sum two-person games on the unit square. In the main part of the paper (Part I) the author studies games whose kernel L is given by: $L(x, y) = K(x, y)$ for $x<y$; $=\phi(x)$ for $x=y$; $=M(x, y)$ for $x>y$ where K and M are defined on the closed triangles $0\leq x\leq y\leq 1$ and $0\leq y\leq x\leq 1$ respectively, possess there continuous second-order derivatives, and the first-order derivatives satisfy $K_x(x, y)>0$, $M_x(x, y)>0$ for $x<1$ and $K_y(x, y)<0$, $M_y(x, y)<0$ for $y<1$; here $\phi(x)$ is a bounded function subject only to the conditions that $\phi(0)$ lie between $K(0, 0)$ and $M(0, 0)$ and $\phi(1)$ lie between $K(1, 1)$ and $M(1, 1)$. The main theorem, whose proof depends on the study of integral equations associated with the game, asserts that the optimal strategies for both players are unique, and consist of jumps at 0 and 1 and a density over some interval. Any one or two of the above components may vanish and a complete enumeration of the various types of optimal strategies and their dependence on L is given. Part II studies kernels which are concave on each side of the diagonal and continuous across it. *A. Dvoretzky.*

HAVE ***Shiffman, Max. Games of timing.** Contributions to the theory of games, vol. 2, pp. 97–123. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J. 1953. $4.00.

The author studies zero-run two-person games on the unit square whose kernel $K(x, y)$ is skew-symmetric, a strictly increasing function of x and a strictly decreasing function

of y in $0 \leq x < y \leq 1$ and, furthermore, $K(x, y)$ can be extended continuously over the closed triangle $0 \leq x \leq y \leq 1$ and this extension has continuous first-order derivatives and, moreover, the set of points where $K_x(x, y) = 0$ or $K_y(x, y) = 0$ contains no linear intervals $x = \text{const}$. If $K(1-, 1) \leq 0$ there is an optimal pure strategy at 1; if $K(0, 1) \geq 0$ there is an optimal pure strategy at 0. Aside from these trivial cases there is a unique optimal strategy which consists of a jump at 0 and a density from some point a to 1 (the jump may occasionally vanish). The proof depends on a study, interesting in itself, of certain integral equations with positive kernel. It goes beyond a mere existence proof and shows how the determination of the optimal strategy depends on the solution of an integral equation. *A. Dvoretsky.*

Glicksberg, I., and Gross, O. **Notes on games over the square.** Contributions to the theory of games, vol. 2, pp. 173–182. Annals of Mathematics Studies, no. 28, Princeton University Press, Princeton, N. J., 1953. $4.00.

Let $K(x, y)$ be the kernel of a zero-sum two-person game on the unit square. (1) A rational K is exhibited for which neither player has an optimal strategy represented by a step-function; this contrasts with the state of affairs when K is a polynomial. (2) A rational K is exhibited for which the optimal strategies of both players are unique and based on a countable set of points dense in $(0, 1)$. (3) For any given strategies f and g there is constructed a kernel in $C^{(\infty)}$ for which f and g are the unique optimal strategies of players 1 and 2 respectively. *A. Dvoretsky* (New York, N. Y.).

Thomsen, Poul. The mathematical treatment of a well known two-person game. Mat. Tidsskr. A. 1952, 63–72 (1952). (Danish)

The author analyzes in full detail the game of Nim [see Hardy and Wright, An introduction to the theory of numbers, Oxford, 1938, chapter IX, §8]. *J. M. Danskin.*

Allais, M. Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école americaine. Econometrica **21**, 503–546 (1953).

The problem of the behavior of a rational man in the face of risk can be described, for the purpose of this review, as that of constructing the most specific theory that can reasonably be defended about how a "rational man" would, or should, choose among tickets in a lottery with possible prizes x, whose probability structure is known to him.

The members of what the author calls the American School (listing them in fn. 66 as M. Friedman, J. Marschak, O. Morgenstern, J. von Neumann, P. Samuelson, and the reviewer) believe, with possible differences of nuance, that much can be said for what may here be called the utility theory; this belief is in fact shared by many others and is not confined to America. The utility theory says simply that for each rational man there is at least one function $u(x)$, attaching a real number (called the utility) to each prize x, such that the man, when confronted with a choice, chooses an alternative for which u has the highest available expected value. It is, I think, universally conceded that if there is such a function, all others are increasing linear functions of it.

The utility theory has its origin in a paper of Daniel Bernoulli [Specimen theoriae novae de mensura sortis, Commentarii academiae scientiarum imperialis Petropolitanae **5** (1730–1731), 175–192 (1738); translated by Alfred Pringsheim as Grundlagen der modernen Wertlehre: Daniel Bernoulli, Versuch einer neuen Theorie der Wertbestimmung von Glücksfällen, Duncker und Humblot, Leipzig, 1896]. It was given a more modern basis by F. Ramsey [The foundations of mathematics and other logical essays, Paul, Trench, and Trubner, London, 1931, pp. 156–198, 199–211]. Current work on the theory is all closely bound to the treatment by von Neumann and Morgenstern as given in the first edition (1944), and filled out in the second edition (1947), of their book "Theory of games and economic behavior" [Princeton; these Rev. **6**, 235; **9**, 50].

The author maintains, and claims to have shown by a questionnaire that the utility theory is not even roughly true of typical people, such as professional men and scholars, who have the reputation of being rational. He also argues that the various axiom systems adduced by members of the American School with a view to rendering the utility theory intuitively acceptable do not in fact do so.

To come to the bone of contention, suppose the man is confronted with a choice between two tickets in the same lottery. Ticket I entitles him to the prize A and Ticket II to the prize B if a white ball is drawn. Either ticket entitles him to the prize C if a white ball is not drawn. The prizes A, B, and C are themselves tickets in another lottery such that, faced with a choice about that lottery, the man would prefer A to B. Now, can this man "rationally" prefer Ticket II to Ticket I in the first-mentioned lottery? The utility theory implies that he cannot. Modern adherents of the utility theory typically either accept this conclusion as appealing, if not self-evident, or they see it as implied by other conclusions that they do accept as appealing. The author, and most others who reject the utility theory, reject the conclusion that Ticket II cannot reasonably be preferred to Ticket I and regard this conclusion as the central defect of the utility theory. *L. J. Savage.*

Mathematical Biology

Geiringer, Hilda. Einige Probleme Mendelscher Genetik. Z. Angew. Math. Mech. **33**, 130–138 (1953).

"Einige Arbeiten, die mich durch mehrere Jahre beschäftigten, sind in mehrfacher Weise zusammengehörig: Sie gehören dem Gebiet der Populationsgenetik an; sie basieren auf Mendels Theorie; . . . schliesslich ist methodologisch jedesmal das gleiche Untersuchungsschema befolgt: Nach Definition der Grundverteilungen werden Rekursionsbeziehungen für gewisse Verteilungen in aufeinander folgenden Generationen gewonnen, diese werden sodann integriert, und es werden insbesondere die stabilen Zustände untersucht, was zur Aufstellung von Grenzwertsätzen führt. Es ist Ziel dieses Artikels aus dieser Gruppe von Untersuchungen einiges zu berichten." (Author's summary.)

The author summarises seven of her papers. Five of these have been reviewed previously [these Rev. **6**, 11; **7**, 319; **12**, 38, 39]. The two papers not previously reviewed appeared in Genetics **33**, 548–564 (1948); **34**, 665–684 (1949). *D. G. Kendall* (Oxford).

Shimbel, Alfonso. Structural parameters of communication networks. Bull. Math. Biophys. **15**, 501–507 (1953).

In continuation of a previous paper [same Bull. **13**, 165–178 (1951); these Rev. **13**, 371] the author defines the "distance" $l(i, j)$ from i to j as the number of links in the shortest

chain from i to j; the "accessibility" $A(i, S)=\sum_j l(i, j)$ of the net S to i; the "accessibility" $A^{-1}(j, S)=\sum_i l(i, j)$ of j to S; and the "dispersion" $D(S)=\sum_i A(i, S)$. He then obtains formulas for the dispersion and for the mean path length of a net formed by joining a site of a net M to a site of a net N by a single symmetric link; and of a net formed by joining a site of each of n nets N_i by a symmetric link to a single central site.

Unfortunately for the reader, the author, having defined the scalar $D(S)$, suddenly speaks of "The square array $D(S)$" (at the bottom of p. 502) without saying that he means now the array of $l(i, j)$. Also on p. 503 he speaks of "adding all of the *zeros* of the ith row and the jth column of *all* of the matrices S^0 through S^T" (author's italics) when he means the number of matrices of the series in each of which the element in the ith row and jth column is zero.

A. S. Householder (Oak Ridge, Tenn.).

Rapoport, Anatol. Spread of information through a population with socio-structural bias. I. Assumption of transitivity. Bull. Math. Biophys. 15, 523–533 (1953).

The author had previously [same Bull. 13, 85–91 (1951); these Rev. 12, 843] obtained a recursion for $P(t)$, the fraction of a population hearing a rumor at t removes from the source, on the assumption that the probability of contact of a pair of individuals is the same for all and constant in time. Here the model is complicated by supposing that the contacts of individuals who are themselves in contact are overlapping. The sum in expression (15) in the paper should be a product, and the page reference to the paper cited above is incorrect. *A. S. Householder.*

Rapoport, Anatol. Spread of information through a population with socio-structural bias. II. Various models with partial transitivity. Bull. Math. Biophys. 15, 535–546 (1953).

The assumption of transitivity treated in part I is modified in various ways to describe an information-diffusion process, in which a certain amount of randomness of contact does occur. In one model a parameter is introduced which is indicative of a tendency to go beyond one's immediate vicinity to spread the information as the vicinity becomes saturated with knowers. In another model the randomness appears in the assumption that new knowers are uniformly distributed among the knowers. Two of the equations thus derived, each with two free parameters are in good agreement with experimental results. (Author's abstract.)

A. S. Householder (Oak Ridge, Tenn.).

TOPOLOGY

Alekszandrov, P. Sz. The notion of space in topology. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 173–188 (1953). (Hungarian)

Lecture given 15 Dec. 1952 at the János Bolyai Mathematics Club's celebration of the 150th anniversay of Bolyai's birth.

Shirota, Taira. The space of pseudo-metrics on a complete uniform space. Osaka Math. J. 5, 147–153 (1953).

Let X be a uniform space and let $SM(X)$ be the set of all bounded pseudo-metrics on X compatible with its uniformity, with distance given by $(\rho, \sigma)=\sup |\rho(x, y)-\sigma(x, y)|$. Then $SM(X)$ is a complete metric space, and when considered as a set of functions on $X\times X$ is a lattice-ordered semi-group. Let X_ρ be the metrizable uniform space on which ρ is an obvious metric, and let $D(X)$ be the set of all such spaces partially ordered in a natural way. Theorem. If X is a complete uniform space then $D(X)$ determines X. From this result the author proves that, for complete spaces, $SM(X)$ considered as either a metric space, or as a lattice ordered semi-group determines X. Here the proofs depend on reconstructing $D(X)$. *M. E. Shanks.*

Pereira Coelho, R. Regularity types. Portugaliae Math. 12, 87–98 (1953).

Let X be a topological space; for $A\subset X$, the closure of A is denoted by A^- and the complement of A by A'. If $A,B\subset X$ and $A^-\subset B'^{-\prime}$, then A is said to be regularly contained in B. This notion was introduced by M. H. Stone and applied to the definition of completely regular spaces [Trans. Amer. Math. Soc. 41, 375–481 (1937)]. Let T be a partially ordered set. X is said to have regularity type T at $x \in X$ if for every open set V_0 containing x, there exists a family $\{V_t\}_{t\in T}$ of open subsets of V_0 all containing x and such that if $t<t'$ in T, then $V_{t'}$ is regularly contained in V_t. A number of very simple theorems concerning this concept are proved. For an arbitrary limit ordinal number μ, an example of a Hausdorff space is constructed which has regularity type ω^μ at every point but does not have regularity type $\omega^{\mu+1}$ at a certain point. *E. Hewitt* (Seattle, Wash.).

Králik, Dezső. Concerning a remark on universal spaces. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 561–562 (1953). (Hungarian)

The remark is that the inverse of a homeomorphism from a separable metric space R into the Hilbert cube is uniformly continuous only if R is totally bounded. *P. R. Halmos.*

Valle Flores, Enrique. A property of Busemann's metric for the subspaces of an arbitrary metric space. Bol. Soc. Mat. Mexicana 10, 71–75 (1953). (Spanish)

The purpose of the note is to prove that a sequence of subsets of a metric space that is a Cauchy sequence in the sense of the set-distance introduced by Busemann [Trans. Amer. Math. Soc. 56, 200–274 (1944); these Rev. 6, 97] converges topologically. *L. M. Blumenthal.*

Floyd, E. E. Orbit spaces of finite transformation groups. I. Duke Math. J. 20, 563–567 (1953).

Let X_G be the orbit space of $\{X, G\}$ where X is a finite simplicial complex and G a finite group operating simplicially on X. The author shows that (1) if X is acyclic over the integers, so is X_G, and (2) if X is an absolute retract, so is X_G. The proof passes from the case in which G is solvable —treated in an earlier paper [Amer. J. Math. 73, 363–367 (1951); these Rev. 12, 846]—to the general case through the use of natural mappings (the so-called transfers) of integral chains $C_n(X_H)\to C_n(X_G)$, H being an arbitrary subgroup of G. *P. A. Smith* (New York, N. Y.).

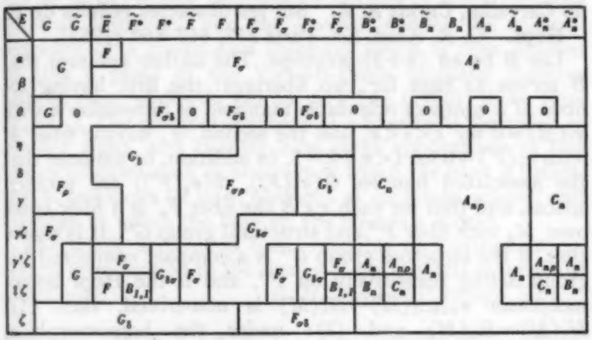
Homma, Tatsuo, and Terasaka, Hidetaka. On the structure of the plane translation of Brouwer. Osaka Math. J. 5, 233–266 (1953).

This paper contains a complete structure analysis of generalized plane translations, i.e. orientation preserving fixed-point free automorphisms of E_2. The main results were

announced in an earlier note [Proc. Japan Acad. **29**, 13–16 (1953); these Rev. **14**, 1003, q.v. for a statement of the principal theorem and definitions involved]. The closed set F which remains after the open sets O_n, O'_n of regular points of a generalized translation are removed from E_2 is the union of at most countably many "singular lines". Much of the paper is devoted to an analysis of the properties of the singular lines. No use is made of the so-called translation theorem of Brouwer. The authors have included a variety of interesting examples. *P. A. Smith.*

Watanabe, Hideaki. L'uniformisation et la séparabilité des ensembles plans. I. Théorèmes fondamentaux. Tôhoku Math. J. (2) **4**, 38–48 (1952).

The principal results of this paper are given in the following table:



where E is a plane set. The notations used are either the usual ones or defined in this note. A point (x_0, y_0), for example, is said to be superior α to E if

$$E_2(P|x=x_0)\cap E_2(P|y\leqq y_0)\cap E\neq 0$$

(0 is the empty set, $E_2(P|L)$ is the set of all points P in the plane with property L). The set of all points of the plane which are superior α to E is designated by αE. βE, γE, δE, etc. are defined similarly. *H. Tong* (New York, N. Y.).

Watanabe, Hideaki. L'uniformisation et la séparabilité des ensembles plans. II. Applications aux problèmes de l'uniformisation et de la séparabilité. Tôhoku Math. J. (2) **4**, 257–263 (1952).

With the help of the table given in the preceding review, the author obtains, among others, the following theorems on uniformization and separability of plane sets which are important in the theory of analytic sets: every plane set (F_σ) $[(F_\sigma{}^*)]$ can be uniformized by a set $(F_{\sigma\delta})$ $[(G_\delta)]$ [this improves certain results of S. Braun, Fund. Math. **28**, 214–218 (1936)]; every plane set (G) can be uniformized by a set $(B^*{}_{1,1})$; if M and N are uniform plane sets such that $M\subset\gamma N$, $\updownarrow M=\updownarrow N$, $\updownarrow M\,\varepsilon\,(B_n)[(A_n)$, (F_σ), (G_δ), (G), $(F)]$, and M, N non-ε $(B_n{}^*)[(A_n)$, $(F_{\sigma\delta})$, (G_δ), (F_σ), $(F)]$, then there does not always exist a plane set

$$S\,\varepsilon\,(A_n)[(A_n),\ (F_\sigma),\ (F_\sigma),\ (G),\ (G)]$$

such that $M\subset\delta S$, $N\subset\delta' S$ (as a corollary of this theorem a negative answer to a problem of Lusin [Mathematica, Cluj **10**, 70–80 (1935)] is obtained); if M and $N\subset\gamma M$ are curves not belonging to $(F^{2\xi})[(G^{2\xi+1})$, $(B_n)]$ $(0\leqq\xi<\Omega)$, then there do not always exist sets S and T such that $M\subset S$, $N\subset T$, $T\subset\gamma S$ and S, $T\,\varepsilon\,(F^{2\xi})[(G^{2\xi+1})$, $(A_n)]$; and if there exist plane sets $M\,\varepsilon\,(\tilde{F}_\sigma)[(\tilde{B}_1)$ and semi-uniform] and $N\,\varepsilon\,(F_\sigma)[(B_1)$ and semi-uniform] such that

$$\updownarrow\{(\gamma M)\cdot(\gamma' N)\}=(\updownarrow M)\cdot(\updownarrow N),$$

then there exists a curve $L\,\varepsilon\,(B_1)$ such that $M\subset\gamma' L$, $N\subset\gamma L$. *H. Tong* (New York, N. Y.).

Al'ber, S. I. Homologies of a space of planes and their application to the calculus of variations. Doklady Akad. Nauk SSSR (N.S.) **91**, 1237–1240 (1953). (Russian)

The author studies the topological nature of the manifold $G(n)$ of the great circles on the n-sphere, or what amounts to the same, of the planes of Euclidean $(n+1)$-space. He obtains, in particular, an estimate of the Frolov-El'sgol'c "length" of $G(n)$ and deduces that on a thrice differentiable Riemann manifold homeomorphic to the n-sphere, there exist at least $n+m-1$ different closed geodesies, where m is the highest power of 2 not exceeding n. The meaning of symbols, such as $R(1-t)$ on the first page, has to be guessed. *L. C. Young* (Madison, Wis.).

Griffiths, H. B. Local topological invariants. Proc. London Math. Soc. (3) **3**, 350–367 (1953).

This is the first of a series of papers to be devoted to "homotopy manifolds," homotopy analogues of the generalized manifolds investigated by Wilder [Topology of manifolds, Amer. Math. Soc. Colloq. Publ., v. 32, New York, 1949; these Rev. **10**, 614]. The author first shows, by an example, that the local Betti groups used by Wilder do not form a good model for a definition of local homotopy groups. Instead, he defines the local r-dimensional homotopy group of the space M at the point x, $C_h{}^r(x)$, as follows: M has the local group $C_h{}^r(x)$ if and only if there is a neighborhood U_Γ^r of x and a function $\Gamma^r(U)$ (whose arguments and values are neighborhoods of x) such that, given U_1 and U_2 satisfying $U_1\subset U_\Gamma^r$, $U_2\subset\Gamma^r(U_1)$, U_2 arcwise connected, then $\pi_r(\bar{U}_2-x|\bar{U}_1-x)\approx C_h{}^r(x)$. [Here $\pi_r(A|B)$ is the image of the injection of $\pi_r(A)$ in $\pi_r(B)$.]

This definition is shown to be meaningful if M is LC^1 and is 0-deletable at x, where the latter is defined as follows: M is 0-deletable at x if and only if there exist functions $\Delta^0(U)$, $\Delta^0(U_1, U_2)$ at x such that, given U_1 and U_2 satisfying $\bar{U}_2\subset\Delta^0(U_1)$, then any two points in the closure of $(\Delta^0(U_1)-U_2)$ can be joined by a path in $\bar{U}_1-\Delta^0(U_1, U_2)$.

Local homology groups, $C_v{}^r(x)$, are defined by replacing π_r in the definition of $C_h{}^r(x)$ by $H_v{}^r$, where the latter stands for the r-dimensional homology group based on Vietoris cycles with compact carriers. Integer coefficients are used. In the last section of this paper a "Hurewicz" theorem is proved: If M is LC^n $(n\geqq 2)$ and if $C_h{}^1(x)=C_v{}^r(x)=0$, $0\leqq r<n$, then a finitely generated group $C_h{}^n(x)$ exists if and only if $C_v{}^n(x)$ exists and is finitely generated, and in this case $C_v{}^n(x)\approx C_h{}^n(x)$. [A slightly weaker result holds if $n=1$.] *E. G. Begle* (New Haven, Conn.).

✓ ***Reidemeister, Kurt. Topologie der Polyeder und kombinatorische Topologie der Komplexe.** 2te Aufl. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Bd 17. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1953. xi+196 pp. DM 14.

Except for a few minor changes, this is the same as the first edition, published in 1938. *E. G. Begle.*

Sugawara, Masahiro. On the homotopy groups of rotation groups. Math. J. Okayama Univ. **3**, 11–21 (1953).

The author deduces, from results of Serre and Toda on homotopy groups of spheres, the values of certain homotopy

groups of rotation groups. These results have all been announced by Serre [C. R. Acad. Sci. Paris **236**, 2475–2477 (1953); these Rev. **14**, 1110]. The calculations are then used to find certain non-zero elements of homotopy groups of spheres. All the results of this type follow from Serre's work [see the paper cited above], with the exception of the elements in $\pi_{16}(S^7)$ and $\pi_{16}(S^4)$, which were obtained by the reviewer [Proc. Cambridge Philos. Soc. **48**, 547–554 (1952); these Rev. **14**, 306]. *P. J. Hilton.*

Shiraiwa, Kenichi. On the homotopy type of an A_n^3-polyhedron ($n\geqq 3$). Amer. J. Math. **76**, 235–251 (1954).

An A_n^p-polyhedron is a finite $(n-1)$-connected polyhedron of at most $n+p$ dimensions. J. H. C. Whitehead has described [Ann. Soc. Polon. Math. **21**, 176–186 (1949); these Rev. **11**, 48] the cohomology system of an A_n^2-polyhedron ($n>2$; he had treated the case $n=2$ elsewhere) and proved (a) that any abstractly defined A_n^2-cohomology system is properly isomorphic to the cohomology system of some A_n^2-polyhedron, (b) that two A_n^2-polyhedra are of the same homotopy type if and only if their cohomology systems are properly isomorphic, and (c) that a homomorphism of the cohomology system of L into that of K may be realized by a map of K into L.

In the present paper the author defines an A_3^3-cohomology system and an A_n^3-cohomology system ($n>3$), and proves results analogous to (a), (b), (c) above. An A_n^3-cohomology system consists of the cohomology groups with integer coefficients, the cohomology groups mod 2 and the cohomology groups mod 4, the Δ and μ operations used by Whitehead, the Steenrod square, Sq^2, and a modified form of Adem's operation [Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952); these Rev. **14**, 306]. The A_3^3-system is a little more complicated. The methods used are based on those of Whitehead; in particular, a reduced A_n^3-complex is defined, whose $(n+2)$-section is taken to be in the Chang normal form [Proc. Roy. Soc. London. Ser. A. **202**, 253–263 (1950); these Rev. **12**, 120]. *P. J. Hilton.*

Blakers, A. L., and Massey, W. S. The homotopy groups of a triad. III. Ann. of Math. (2) **58**, 409–417 (1953).

[For parts I–II see Ann. of Math. (2) **53**, 161–205 (1951); **55**, 192–201 (1952); these Rev. **12**, 435; **13**, 485.] Let X be a CW complex, and A, B, subcomplexes. It is shown that if $m, n\geqq 3$, if the pair $(A, A\cap B)$ is $(m-1)$-connected, and if $(B, A\cap B)$ is $(n-1)$-connected, then the first non-vanishing triad group $\pi_{m+n-1}(X; A, B)$ is isomorphic to the tensor product $\pi_m(A, A\cap B)\otimes\pi_n(B, A\cap B)$ the isomorphism being realized by a generalized Whitehead product. This result is applied to show that if X is m-connected for some m, $1<m<n-2$, and if X^* is obtained from X by adjoining any collection of n-cells having a point in common, then the Whitehead product gives an isomorphism of $\pi_{m+1}(X)\otimes\pi_n(X^*, X)$ into $\pi_{m+n}(X^*, X)$ and the composition operation, an isomorphism of $\pi_n(X^*, X)\otimes\pi_{m+n}(I^n, \dot{I}^n)$ into $\pi_{m+n}(X^*, X)$; further, $\pi_{m+n}(X^*, X)$ is the direct sum of these two image groups. The structure of $\pi_{2n-2}(X^*, X)$ is also determined. These results apply to give the structure of the homotopy groups of a union of n-spheres with a single point in common, for certain dimensions. It follows from this theorem, that any universally defined $m$ary homotopy construction from dimension n to a dimension $\leqq 3n-2$ can be obtained by iterated use of Whitehead products and the composition operation. *J. Dugundji.*

Kervaire, Michel. Extension d'un théorème de G. de Rham et expression de l'invariant de Hopf par une intégrale. C. R. Acad. Sci. Paris **237**, 1486–1488 (1953).

Let A^p, B^q be differential forms of degrees p, q respectively on a differentiable manifold X_n of dimension n and let a^p, b^q be their corresponding singular cochains. It is proved that a necessary and sufficient condition such that

$$\int_{c_r} A^p\wedge B^q = a^p\cup b^q(c_r),\quad r=p+q,$$

for any finite differentiable chain c_r in X_n is that

$$a^p\cup b^q(c_r)=a^p\cup b^q(Uc_r)$$

for any subdivision Uc_r of c_r. From this an integral formula is derived for the Hopf invariant of a continuous mapping of a $(2n-1)$-sphere into an n-sphere. *S. Chern.*

de Carvalho, Carlos A. A. Sur les obstacles réduits de H. Hopf. C. R. Acad. Sci. Paris **237**, 867–869 (1953).

Let B be an $(n+k)$-polytope. The author assumes that B serves as base for two fiberings: the first having for fiber M a compact orientable manifold of dimension n with $\pi_i(M)=0$ for $1<i<k$, and the second, $\mathfrak{F}'$, having fiber F' with $\pi_i(F')=0$ for $i<n+k-1$. In addition, he assumes that the associated bundles $\mathcal{B}(\pi_q(F))$, $\mathcal{B}(\pi_q(F'))$ are product spaces, and that for each $x\in B$ the fiber F_x' is a fiber space over M_x with fiber F'' and structural group G''. It is shown that, if the structure group G'' is a compact connected Lie group acting transitively on F'', and if the Hopf homomorphism $\pi_{n+k-1}(M)\to H_k(M)$ is non-trivial, then (1): $H_k(M)\approx H_n(M)$ and (2): under the homomorphism $C^{n+k}(B, \pi_{n+k-1}(F'))\to C^{n+k}(B, H_k(M))$ induced by the projection homomorphism followed by Hopf's homomorphism, the obstruction to an $(n+k)$-cross-section in $\mathfrak{F}'$ has the same image as the Whitney-Steenrod characteristic class of the fibering $\mathfrak{F}'$. *J. Dugundji* (Los Angeles, Calif.).

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***Borel, A. Cohomologie des espaces localement compacts, d'après J. Leray.** Séminaire de topologie algébrique, printemps 1951. Ecole Polytechnique Fédérale, Zurich, 1953. 95 pp. (miméographié). 7.00 francs suisses.

This work consists of mimeographed reports of a seminar on algebraic topology in Zurich during the spring of 1951. The material covered is the theory of homological invariants of locally compact spaces and of continuous mappings based on the approach used by Leray [J. Math. Pures Appl. (9) **29**, 1–80, 81–139 (1950); these Rev. **12**, 272]. Altogether there are nine chapters of which the first five are devoted to the cohomology theory of locally compact spaces, first with constant coefficients and later with a sheaf (faisceaux) of coefficients, and the last four develop the spectral sequence of a continuous map and apply it to the theory of fiber spaces.

The first chapter is concerned with algebraic properties of modules and algebras (which may have various additional structures such as a gradation or a differential operator). The second chapter introduces the concept of a "complex" on a space X. This is not to be confused with the usual use of the term in algebraic topology as it denotes a module together with a support function from the module to closed subsets of X, and this function is required to satisfy suitable conditions depending on the algebraic structures present in the module. Two important special types of complexes are introduced. One is a fine complex and the other is a couverture. The former is a complex generated in a certain sense

by elements with small support, and the latter is a graded differential complex without torsion in which the local homology groups are trivial in dimensions different from zero and isomorphic to the coefficient ring in dimension zero. The main result of the third chapter is the uniqueness theorem asserting that in any two fine couvertures the elements with compact supports have isomorphic homology modules (algebras if they are algebras). This proof makes no use of spectral sequences but is based on inductive arguments in suitable tensor products. The fourth chapter applies the uniqueness theorem to get the de Rham theorems and relations between the singular groups and the Čech groups in spaces satisfying enough local-connectedness assumptions. Generalized coefficients (faisceaux) are introduced in chapter five and a uniqueness theorem for homology with coefficients in such a system is established.

The sixth chapter contains definitions of filtrations and spectral sequences, and the seventh defines the spectral sequence of a continuous mapping which is then applied to give a proof of the Vietoris-Begle mapping theorem [Begle, Ann. of Math. (2) **51**, 534–543 (1950); these Rev. **11**, 677]. The spectral sequence of a fibering is discussed in chapter eight, and chapter nine gives applications of this to special cases such as fiberings in which the fiber is totally non-homologous to zero, fiberings of euclidean spaces, and fiberings in which the base on the fiber is a sphere.

The whole account is quite readable and makes an excellent introduction to the theory of the spectral sequence of a mapping. *E. H. Spanier* (Chicago, Ill.).

Nakaoka, Minoru. Note on cohomological operations. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **4**, 51–58 (1953).

The author gives an axiomatic theory of $\curlyvee$-operations in finite, augmentable cell-complexes. He proves explicitly the existence of such operations and the uniqueness of the resulting Steenrod squares, Pontryagin squares, Postnikov squares, and cup-products of cohomology classes. The Cartan product formula [C. R. Acad. Sci. Paris **230**, 425–427 (1950); these Rev. **12**, 42] for Steenrod squares is recovered, and product formulae for the Pontryagin and Postnikov squares obtained. Let J_i be the homomorphism

$$H^p(K, L; Z_{2k})\to H^{2p-i}(K, L; Z),$$

$(p-i)$ odd, defined by Shimada and Uehara [Nagoya Math. J. **4**, 43–50 (1952); these Rev. **13**, 859]. The author shows that $J_i=\Delta \mathrm{Sq}_{i+1}$, where Δ is the Bockstein operator

$$\Delta:\quad H^{2p-i-1}(K, L; Z_2)\to H^{2p-i}(K, L; Z).$$

P. J. Hilton (Cambridge, England).

Eckmann, Beno. On complexes with operators. Proc. Nat. Acad. Sci. U. S. A. **39**, 35–42 (1953).

In this paper several general theorems are established which extend earlier results of the author, Eilenberg, and others. An outline of various applications of these theorems is given.

Let R be a ring with a unit, and C an R-complex. Further let J be an abelian group, Φ the R-module of additive homomorphisms of R into J, and Ψ a submodule of Φ. The author shows how to choose a subcomplex of the cochain complex of C with coefficients in J in a manner which depends only on the choice of Ψ. The cohomology groups of this subcomplex are denoted by $H_\Psi{}^p(C, J)$. The first main theorem asserts that $H_\Psi{}^p(C, J)$ is naturally isomorphic with $H^p(C, \Psi)$.

Now let A be a group, R_A the group ring of A with integer coefficients, and J an R_A-module. The second main theorem asserts that J may be extended to an R_A-module J' such that $H^p(A, J')=0$ for all $p>0$ [R. C. Lyndon, Ann. of Math. (2) **52**, 650–665 (1950), p. 653; these Rev. **13**, 819]. This theorem, though straightforward, is important in the cohomology theory of groups because it allows dimension shifting, and a uniqueness proof by the standard Eilenberg technique. Let B be a subgroup of A, and Ψ the submodule of Φ such that $\psi\in\Psi$ if and only if $\psi(ba)=b\psi(a)$ for all $b\in B$, $a\in A$. Theorem 3 now says that $H^p(B, J)$ is naturally isomorphic with $H^p(A, \Psi)$ for all p [A. Weil, J. Math. Soc. Japan **3**, 1–35 (1951); these Rev. **13**, 459]. Theorem 4 is the topological analogue of Theorem 3. It relates the cohomology of a space and a covering space.

After devoting a section to the study of the transfer operation in the cohomology theory of groups and covering spaces, the paper concludes with a theorem relating the derivatives in a group [R. H. Fox, Ann. of Math. **57**, 547–560 (1953); these Rev. **14**, 843] with the number of ends of the group [H. Hopf, Comment. Math. Helv. **16**, 81–100 (1944); these Rev. **5**, 272]. *J. C. Moore.*

Gamkrelidze, R. V. Computation of the Chern cycles of algebraic manifolds. Doklady Akad. Nauk SSSR (N.S.) **90**, 719–722 (1953). (Russian)

The following theorem is stated: Let M^k be an algebraic manifold in complex projective n-space P^n; let $Q=P^{n-k-2+t}$ and $P^{n-s+t}\supset Q$ be two fixed subspaces of P^n; denote the intersection $M^k\cap P^{n-s+t}$ by M^{k-s+t}. The points $z\in M^{k-s+t}$, for which there exists a $P^{n-k-1+t}$, containing z and Q, the dimension of whose intersection with the tangent space to M^{k-s+t} at z is $\geqq t$, form a $2(k-s)$-cycle Π_t^{k-s} on M^k. Then the Chern cycle Γ^{k-s} of dimension $2(k-s)$ of M^k is given by

$$\Gamma^{k-s}=\sum_{t=0}^{s}(-1)^t\binom{k-t+1}{k-s+1}\Pi_t^{k-s}.$$

For $s=k$ this specializes to a formula for the characteristic:

$$\chi(M^k)=\sum_{t=0}^{k}(-1)^t(k-t+1)\ \mathrm{Ind}\ (\Pi_t^0);$$

here Ind (Π_t^0) is the class of the intersection M^t of M^k and P^{n-k+t}. It is stated that the formula for Γ^{k-s}, given by E. Kundert [Proc. Nat. Acad. Sci. U. S. A. **38**, 893–895 (1952); these Rev. **14**, 682] is in error, since it omits all terms with $t>0$ of the above formula.

The proof is not given, but material necessary for it is discussed: Let $P^n{}_{k,l}$ be the space of pairs (z, P^k), where z is a point of P^n, P^k a k-subspace of P^n, $z\in P^k$, and moreover z lies in a fixed P^l. The homology of $P^n{}_{k,n}$ is described in terms of the homology of the Grassmann manifold $H(k, n-k)=P^n{}_{k,0}$ (following Ehresmann); there is no torsion, and vanishing homology in all odd dimensions. For every $z\in P^n$ let $E_z{}^n$ be the complex affine space, obtained by removing the polar P^{n-1} of z from P^n. One constructs a fiber bundle $T=T(E^k, P^n{}_{k,n})$ over $P^n{}_{k,n}$, by taking as fiber over the point (z, P^k) of $P^n{}_{k,n}$ the space $P^k\cap E_z{}^n$, which is an affine E^k. The imbedding of M^k into P^n defines a map f of M^k into $P^n{}_{k,n}$ by sending z into the pair (z, tangent plane to M^k at z). The bundle induced by f and T is the tangent bundle of M^k; the Chern classes of T determine therefore those of M^k. The determination of the Chern classes of T is given in outline; the idea is to construct explicitly certain systems of vector fields in T (or rather in the part of T corresponding to any given cycle of $P^n{}_{k,n}$) and to consider their cycles of singularities. *H. Samelson.*

Weyl, Hermann. Über die kombinatorische und kontinuumsmässige Definition der Überschneidungszahl zweier geschlossener Kurven auf einer Fläche. Z. Angew. Math. Physik 4, 471–492 1953.

The author gives the details of definitions of the intersection number (which he calls the characteristic) of two one-dimensional homology classes on an orientable two-dimensional surface, as indicated in his classical book: "Die Idee der Riemannschen Fläche" [Teubner, Leipzig-Berlin, 1913]. The first definition makes use of a triangulation of the surface and utilizes the realization of the homology classes on two dual cell decompositions. As this definition is not symmetrical, a main effort consists in the proof of the anti-symmetry of the intersection number, which is carried out. A second definition is to make use of cohomology theory and works, not with a triangulation, but with a covering of the surface. Its essential idea is the geometrical definition of the intersection number as the number of positive intersections minus the number of negative intersections. These two definitions are proved to be identical. Finally, it is stated that, at least for differentiable surfaces, the use of a triangulation in the study of Riemann surfaces and their functions and integrals is a detour and is not worthwhile. (In fact, the following remark has been prevailing among French mathematicians: "Quand on coupe une surface de Riemann, on la tua.") The author promises to take care of this remark in a new edition of the above-mentioned book, now under preparation. *S. Chern* (Chicago, Ill.).

GEOMETRY

***Tornetta, Filippo. Per una teoria completa dei poligoni.** Società Italiana per il Progresso delle Scienze, XLII riunione, Roma, 1949, Relazioni, Vol. primo, pp. 215–217. Società Italiana per il Progresso delle Scienze, Roma, 1951.

The author draws attention to some elementary theorems on the sum of the angles of a general polygon [cf. M. Brückner, Vielecke und Vielflache, Theorie und Geschichte, Teubner, Leipzig, 1900]. *H. S. M. Coxeter.*

Riedwil, H., und Debrunner, H. Drei neue Näherungskonstruktionen für die Quadratur des Kreises. Elemente der Math. 9, 16–18 (1954).

Swain, R. L. Bounded models of the Euclidean plane. I. Condensed graphs. Amer. Math. Monthly 61, 21–26 (1954).

The author maps the Euclidean xy-plane onto the interior of a square by means of the relations $x'=x/(1+|x|)$, $y'=y/(1+|y|)$. This mapping provides a means for visualizing the behavior of a function $y=f(x)$ for large values of x and y and has therefore some didactic interest. It is shown that the graphs preserve maxima and minima but not inflexion points. Symmetry properties and various applications of the mapping are studied and the essential properties of mapping functions are discussed. *E. Lukacs.*

Gans, David. Bounded models of the Euclidean plane. II. A circular model of the Euclidean plane. Amer. Math. Monthly 61, 26–30 (1954).

The author gives in this paper another [see the preceding review] bounded model of the Euclidean plane. The xy-plane is mapped onto the interior of the unit circle by means of the relations $x'=x(1+x^2+y^2)^{-\frac{1}{2}}$, $y'=y(1+x^2+y^2)^{-\frac{1}{2}}$. This model permits also the introduction of the ideal points needed to convert the Euclidean plane into the plane of projective or of elliptic geometry. *E. Lukacs.*

May, K. O. Bounded models of the Euclidean plane. III. The use of condensed graphs in analytic geometry. Amer. Math. Monthly 61, 31–32 (1954).

The author constructs a special graph paper to make more convenient the use of the circular model of the Euclidean plane described in the paper by D. Gans [see the preceding review]. *E. Lukacs* (Washington, D. C.).

Andress, W. R., and Saddler, W. Perspective triads. With a note by W. W. Sawyer. Math. Gaz. 37, 247–255 (1953).

The first two authors give an analytic treatment of the configuration in the complex projective plane formed by two sextuply perspective triangles. The six centers of perspective form two further triangles such that any two of the four triangles are sextuply perspective, with the other two as centers of perspective. The third author points out that the twelve sides of the four triangles pass by fours through nine points which are the points of inflection of a plane cubic curve. Thus the whole figure consists of a configuration $(12_3, 9_4)$ and its dual $(9_4, 12_3)$ [cf. Veblen and Young, Projective geometry, v. 1, Ginn, Boston, 1910, pp. 100, 247–249]. *H. S. M. Coxeter* (Toronto, Ont.).

Pierce, William A. The impossibility of Fano's configuration in a projective plane with eight points per line. Proc. Amer. Math. Soc. 4, 908–912 (1953).

The author considers a configuration 7_3 (i.e., a complete quadrilateral with concurrent diagonals) in a projective plane with 8 points on each line. Of the $1+7+7^2=57$ points in the plane, 15 do not lie on any line of the 7_3. Each point of the 7_3 lies on 5 lines not belonging to the 7_3, and each of these lines contains 3 of the 15 "extra" points. The 7_3 thus determines 7 ways of distributing the 15 points by 3's on 5 lines, as in the School Girl Problem of T. P. Kirkman [Lady's and Gentleman's Diary 147, 48 (1850)]. By examining all the solutions of Kirkman's problem, as given by F. N. Cole [Bull. Amer. Math. Soc. 28, 435–437 (1922)], the author finds that none can actually be realized in this geometrical manner. He concludes that no projective plane with 57 points admits a configuration 7_3. (By a quaint misprint, he gives Kirkman's date as 1950 instead of 1850.) *H. S. M. Coxeter* (Toronto, Ont.).

Hall, Marshall, Jr. Uniqueness of the projective plane with 57 points. Proc. Amer. Math. Soc. 4, 912–916 (1953).

It was proved independently by R. C. Bose [Sankhyā 3, 323–338 (1938)] and W. L. Stevens [Ann. Eugenics 9, 82–93 (1939)] that the existence of a projective plane with $n+1$ points on each line is equivalent to the existence of a completely orthogonal set of n-by-n Latin squares. For the case $n=7$, the enumeration of Latin squares was carried out with enormous labor by H. W. Norton [Ann. Eugenics 9, 269–307 (1939); these Rev. 1, 199] and A. Sade [Ann. Math. Statistics 22, 306–307 (1951); these Rev. 12, 665]. On the

basis of this enumeration, Bose and Nair [Sankhyā 5, 361–382 (1941); these Rev. 4, 33] proved that $PG(2, 7)$ is the only projective plane with 8 points per line. The author gives a new and much simpler proof of this important theorem. He finds that any non-Desarguesian plane with 57 points would contain a 7_3 configuration, which has been shown to be impossible by W. A. Pierce [see the preceding review]. He remarks that the analogous question of geometries having 9 points per line cannot be settled in such a simple manner. *H. S. M. Coxeter* (Toronto, Ont.).

Di Noi, Salvatore. Le varie metriche del piano proiettivo. Period. Mat. (4) 31, 296–313 (1953).

Lenz, Hanfried. Beispiel einer endlichen projektiven Ebene, in der einige, aber nicht alle Vierecke kollineare Diagonalpunkte haben. Arch. Math. 4, 327–330 (1953).

The configuration of Fano is a quadrilateral whose diagonal points are collinear. It is also the finite plane with seven points. In the Desarguesian case the Fano configuration corresponds to characteristic 2. The author exhibits a finite geometry with $n+1$ points on a line with $n=16$ in which some but not all quadrilaterals have their diagonal points collinear. This geometry is given by a Veblen-Wedderburn system determined by an irreducible quadratic over the field with four elements. [Cf. the reviewer's paper, Trans. Amer. Math. Soc. 54, 229–277 (1943), p. 274; these Rev. 5, 72]. It might be worth noting that this can also happen for $n=9$. By the method of pp. 244–251 of the reviewer's paper there is a geometry given by the doubly transitive group generated by (012)(345)(678), (1326)(4587), and (1428)(3765). Here the points (0, 0), (1, 1) (5, 2), (4, 4), (7, 8), (2, 3), and the infinite point (5) form a Fano configuration, though most quadrilaterals do not have collinear diagonal points. If numbered from 1 to 7 as listed here, points i, $i+1$, and $i+3$ modulo 7 are collinear. *Marshall Hall, Jr.* (Columbus, Ohio).

Sce, Michele. Sugli r birapporti di $r+3$ punti diun S_r. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 363–374 (1952).

Let S_r be a projective r-space on a field γ. Given in S_r $r+3$ points $P_1, \cdots, P_{r+3}$, the author defines in a natural way r elements of γ, called the generalized cross-ratios of the set $P_1, \cdots, P_{r+3}$, which are projective invariants of that set. Let the characteristic of γ be $\neq 2$; if all the generalized cross-ratios belong to the prime subfield of γ (resp., are all equal to a given element α of that subfield), the set $P_1, \cdots, P_{r+3}$ is said to be quasi-harmonic (resp., harmonic of degree α); a collineation of S_r may be characterized as a 1-1 mapping of S_r onto itself which carries quasi-harmonic sets onto quasi-harmonic sets with the same generalized cross-ratios (resp., as a 1-1 mapping of S_r onto itself which preserves the harmonic sets of a given degree α). This generalizes immediately to collineations between distinct spaces.

The author introduces also harmonic sets with respect to a variety V which may be either a normal curve V_1^r or a hyperquadric V^2_{r-1}. The meaning of some arguments in section 6 is not clear to the reviewer; apparently, the last statement of that section is incorrect. *J. L. Tits.*

Lombardo-Radice, Lucio. Una nuova construzione dei piani grafici desarguesiani finiti. Ricerche Mat. 2, 47–57 (1953).

This is another proof of a result originally obtained by Hilbert in his Grundlagen der Geometrie [7. Aufl., Teubner, Leipzig, 1930]. Assuming the affine Theorem of Desargues it is shown that a plane can be coordinatized by a division ring and thus that the full projective Theorem of Desargues is valid. *Marshall Hall, Jr.* (Columbus, Ohio).

Panvini, Jean. Alcune osservazioni sulle geometrie non archimedee. Ann. Scuola Norm. Super. Pisa (3) 7, 153–159 (1953).

Let C_P be the field of functions of a real variable t, obtained from the function t by using the four rational operations and the operation $(1+\omega^2)^{\frac{1}{2}}$, and ordered as follows: $f>\varphi$ if and only if there exists a t_0 such that $f(t)-\varphi(t)>0$ for all $t\geqq t_0$. Making use of C_P, Hilbert [Grundlagen der Geometrie, 7. Aufl., Teubner, Leipzig, 1930, p. 47] has constructed a model of a non-archimedean geometry, i.e. a geometry satisfying all Hilbert axioms for the 3-dimensional euclidean geometry except for the postulate of Archimedes. The author shows that every non-archimedean geometry may be constructed in a similar way from a non-archimedean field C, closed under the operation $(a^2+b^2)^{\frac{1}{2}}$. Moreover, every such field has a subfield isomorphic to C_P; therefore, the most general non-archimedean geometry "contains" Hilbert's model.

In the geometry constructed on C_P, a circle and a straight line, going through its interior and lying in its plane, do not necessarily meet. This remains true when C_P is replaced by a field $\bar{C}_P$, obtained from C_P in a similar (but not identical) way as the one used generally to define the real numbers from the rationals. $\bar{C}_P$ is continuous in the sense of Cantor; therefore, the situation described above may arise even in geometries verifying Cantor's continuity axiom. *J. L. Tits* (Brussels).

Szász, Pál. Various ~~A different~~ elementary presentations of hyperbolic trigonometry. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 209–218 (1953). (Hungarian)

Convex Domains, Extremal Problems, Integral Geometry

Vincensini, Paul. Sur certains ensembles déduits des ensembles d'arcs de cercle ou de calottes sphériques. Bull. Sci. Math. (2) 77, 120–128 (1953).

Repetition and slight extension of an earlier result of the author [C. R. Acad. Sci. Paris 232, 2075–2076 (1951); Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Perrella, Roma, 1953, pp. 456–464; these Rev. 12, 850; 15, 150]. *V. L. Klee* (Seattle, Wash.).

Jarden, Dov. On the impossibility of "cubing" a cube. Riveon Lematematika 7, 79–80 (1954). (Hebrew. English summary)

Same proof as given by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte [Duke Math. J. 7, 312–340 (1940), §10; these Rev. 2, 153] of the theorem: No rectangular parallelepiped in Euclidean n-space, $n\geqq 3$, can be divided into a finite number of pairwise incongruent cubes. *E. G. Straus* (Los Angeles, Calif.).

Hanani, Haim. On a point of minimum sum of distances-squares from the faces of a simplex. Riveon Lematematika 7, 10–12 (1954). (Hebrew. English summary)

Valentine, F. A. Minimal sets of visibility. Proc. Amer. Math. Soc. 4, 917–921 (1953).

With $V\subset S\subset E_n$, V is a set of visibility in S provided for each $p \varepsilon S$ there is a $q \varepsilon V$ such that S contains the line segment pq. Theorem: If S is a compact set in E_2, and for each $x \varepsilon S$ there is a unique set $V(x)$ minimal with respect to $[x \varepsilon V$ and V is a closed convex set of visibility in $S]$, then either S is convex or S is star-shaped from exactly one point of S. Whether the Theorem remains valid with "compact" replaced by "closed", or E_2 by E_n, are mentioned as open questions. There is obtained a result concerning closed $S\subset E_n$ such that for each $x \varepsilon S$ there is a unique $V(x)$ minimal with respect to $[x \varepsilon V$ and V is a continuum of visibility in $S]$. *V. L. Klee* (Seattle, Wash.).

Ohmann, D. Isoperimetrische und verwandte Extremalprobleme für beschränkte ebene Punktmengen. J. Reine Angew. Math. 192, 65–73 (1953).

This paper is anterior to two others already reviewed [Math. Ann. 124, 265–276 (1952); Math. Z. 55, 299–307 (1952); these Rev. 13, 864, 971] which are in the same order of ideas. Given a bounded plane set A, let $q(\xi)$ be the inner (outer) linear measure of the projection of A on a line perpendicular to the unit vector ξ, and let A^* be the convex figure whose support-function is the convex minorant of $q(\xi)/2$. The author's main result is that the inner (outer) plane measure of A^* is not less than that of A. Since $q(\xi)$ and related quantities are not increased by replacing A by A^*, this provides an automatic extension, to bounded plane sets, of certain inequalities known for convex figures. *L. C. Young* (Madison, Wis.).

Grotemeyer, K. P. Gleitverbiegungen und eindeutige Bestimmtheit isometrischer, ebenrandiger Mützen. Math. Z. 59, 278–289 (1953).

Denote by cap a surface in E^3 which is homeomorphic to a disk, has a plane boundary, intersects any line perpendicular to the plane of the boundary at most once, is of class C^3, and has positive Gauss curvature. It is shown that a cap is both rigid and infinitesimally rigid within the family of caps. The rigidity, under much weaker assumptions and with a much more complicated proof, is due to Pogorelov. *H. Busemann* (Los Angeles, Calif.).

Grotemeyer, K. P. Bemerkung zur Verbiegung konvexer Flächen. Math. Z. 59, 258 (1953).

Let F be a surface with positive Gauss curvature of class C^3 bounded by a shadow boundary. Among all surfaces of class C^3 intrinsically isometric to F the surface F is distinguished by the property that the spherical image of the boundary has minimal length $(=2\pi)$. *H. Busemann.*

Algebraic Geometry

Segre, Beniamino. Il contrasto fra continuo e discontinuo e la geometria algebrica. Archimede 5, 221–225 (1953).

Spampinato, Nicolò. Trasformazione birazionale determinata da una ipersuperficie dell'S_r complesso per lo scioglimento delle sue singolarità. Giorn. Mat. Battaglini (5) 1(81), 113–122 (1953).

Spampinato, Nicolò. Effettiva determinazione di una trasformazione birazionale trasformante una data ipersuperficie dell'S_r in una varietà dell'S_{r,t_r} priva di punti multipli. Giorn. Mat. Battaglini (5) 1(81), 123–152 (1953).

Spampinato, Nicolò. Sull'estenzione del teorema di Lüroth dall'S_1 complesso ad un S_1 ipercomplesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 190–193 (1953).

Spampinato, Nicolò. Nozioni introduttive alla teoria delle ipersuperficie algebriche di indice n dell'S_r proiettivo complesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 198–199 (1953).

Balsimelli, Pio. Breve studio di una trasformazione birazionale dell'S_3 complesso determinata da una trasformazione quadratica biduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 171–174 (1953).

Bruno, Rita. Studio di una trasformazione cremoniana dell'S_3 dedotta da una trasformazione quadratica dell'S_2 tripotenziale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 120–124 (1953).

Cirillo, Elda. Sulle rappresentazioni complesse dell'S_2 tripotenziale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 194–197 (1953).

Fadini, Angelo. Studio di una trasformazione cremoniana dell'S_3 dedotta da una trasformazione quadratica dell'S_2 triduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 19 (1952), 115–119 (1953).

Derwidué, L. A propos des involutions cycliques. Bull. Soc. Roy. Sci. Liège 22, 24–28 (1953).

An alternate proof for one of the results of a previous paper [Mém. Soc. Roy. Sci. Liège (4) 11, no. 5 (1951); these Rev. 13, 866]. *R. J. Walker* (Ithaca, N. Y.).

Tibiletti, Cesarina. Costruzioni a priori della sestica con nove cuspidi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 207–220 (1952).

Aprés avoir étudié l'effet de certaines opérations sur les traits canoniques et élémentaires, au sens de l'algèbre des tresses de Dedo, l'auteur part de la forme canonique de la tresse d'une sextique générale et étudie les opérations à effectuer sur les traits de manière à amener des diramations à coïncider pour donner naissance à neuf cuspides. La même tresse est ensuite construite à partir de la forme limite réduite à trois droites doubles. *B. d'Orgeval* (Alger).

Tibiletti, Cesarina. Complementi all'algebra delle trecce caratteristiche e loro applicazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 249–259 (1952).

A l'aide de nouveaux résultats, sur les opérations effectuées sur des traits, dans lesquels s'entrecroisent 2 fils enveloppés par un troisième en ordre inverse des traits élémentaires de Dedo, l'auteur donne une nouvelle forme de la tresse caractéristique de la sextique à neuf cuspides, forme plus symétrique. *B. d'Orgeval* (Alger).

Lévy-Bruhl-Mathieu, Paulette. Etude des surfaces d'ordre $p-2+h$ $(0\leqq h<p)$ passant par une courbe canonique de genre p. Application à la classification des courbes algébriques de genre inférieur à 11. C. R. Acad. Sci. Paris 236, 2032–2034 (1953).

Outline of a method of classifying algebraic curves studying the properties of surfaces containing the canonical model of the curve, and thereby determining the special series contained on the curve. *R. J. Walker.*

Gallarati, Dionisio. Intorno ad una superficie del sesto ordine avente 63 nodi. Boll. Un. Mat. Ital. (3) 7, 392–396 (1952).

If $\varphi(x_0, \cdots, x_4)=0$ is a quadric in S_4, touching the hyperplane faces of the simplex of reference, $\varphi(x_0^2, \cdots, x_4^2)=0$ is a quartic hypersurface with 40 nodes, 8 in each face of the simplex forming a configuration projectively equivalent to the vertices of a cube. The apparent contour of this from any one of its nodes is a sextic surface in S_3 having 39 nodes, projections of the remaining nodes of the quartic, and 24 more, arising from lines on the quartic through the centre of projection, i.e., 63 nodes in all. The last 24 are the intersections of a quadric, a cubic, and a quartic; 32 of the others form 4 configurations each projectively equivalent to the vertices of a cube, and the remaining 7 are coplanar and are the vertices and diagonal points of a complete quadrangle. Comparing these facts with the known properties of the configuration of 63 nodes on a sextic surface due to Segre [Atti Accad. Ligure 9, 15–22 (1953); these Rev. 15, 249], the author tentatively concludes that the surfaces are not the same; but neither configuration has been investigated in sufficient detail for this to be certain.

P. Du Val (Bristol).

Reynolds, J. A variety with a certain singular point. Proc. Cambridge Philos. Soc. 50, 143–144 (1954).

V is an irreducible variety of d dimensions defined over an algebraically closed ground field of characteristic zero. A point P of V is simple if and only if the ideal of non-units in the quotient ring of the point has a basis of d elements. The author considers a point P of V such that its quotient ring is integrally closed, and the ideal of non-units in the quotient ring has a basis of $d+1$ elements. He shows that there is a point P' in regular correspondence with P on a primal V' which is birationally equivalent to V.

D. Pedoe (Khartoum).

Marchionna, Ermanno. Curve e varietà di diramazione per superficie ed ipersuperficie multiple generali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 473–483 (1952).

Define a "general" μ-ple S_{r-1} as the projection (from a point not lying on it) of a general μ-ic hypersurface in S_r, and a "general" μ-ple hypersurface in S_{r-1} as the section by a hypersurface of the "general" μ-ple S_{r-1}. The following theorem is proved. An irreducible variety Γ of $r-3$ dimensions lying on a hypersurface f_m of order m in S_{r-1} is the branch locus of a determinate "general" μ-ple f_m, if: (i) the order of Γ is $n=m\mu(\mu-1)$; (ii) Γ has a cuspidal locus of $r-4$ dimensions and order $m\mu(\mu-1)(\mu-2)$; (iii) Γ contains an $(r-4)$-dimensional variety R_0, coresidual to a hyperplane section, which is the complete intersection of three hyperplanes of orders $m, \mu, \mu-1$; (iv) for $r\geqq 5$, R_0 is irreducible and free from multiple loci of $r-5$ dimensions; for $r=4$, the set of points R_0 belongs to a g_n^1 not having any double point of the curve Γ as neutral pair. *P. Du Val.*

Marchionna, Ermanno. Varietà intersezioni complete e varietà di diramazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 82–102 (1952).

It is first proved that either of the following two conditions is necessary and sufficient for a non-singular variety Γ^* of $r-2$ dimensions and order $\mu\nu$ in S_r to be the complete intersections of two hypersurfaces of orders μ, ν: (i) The projection of Γ^* from a general point is a hypersurface in S_{r-1} having an $(r-3)$-dimensional double locus of order $\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ which is its complete intersection with a hypersurface of order $(\mu-1)(\nu-1)$; (ii) hypersurfaces of order $\mu+\nu-r-1$ in S_r trace on Γ^* varieties of its canonical system.

Next the following theorem of Jongmans [Mém. Soc. Roy. Sci. Liège (4) 7, 367–468 (1947); these Rev. 9, 611] is proved in detail: A variety V_{r-d} of S_r, irreducible and without an $(r-d-1)$-dimensional multiple locus, is the complete intersection of d hypersurfaces of orders $m_1, \cdots, x_d$, if the same is true of its general hyperplane section; with the further conditions (for $d=r-1$ only, i.e. in the case in which V is a curve) that its genus is $p=\frac{1}{2}n(m_1+\cdots+m_d-r-1)$ (where $n=m_1\cdots m_d$ is its order).

The last part of the paper is devoted to the case in which the intersecting hypersurfaces have a singular point in common, but the theorems arrived at are too lengthy to quote. *P. Du Val* (Bristol).

Marchionna, Ermanno. Sulle proiezioni delle varietà intersezioni complete di due ipersuperficie. Boll. Un. Mat. Ital. (3) 8, 265–268 (1953).

As a complement to result (i) of the paper reviewed above, the necessary and sufficient conditions for a hypersurface Γ of order $\mu\nu$ in S_{r-1} to be the projection of the complete non-singular intersection of two hypersurfaces of orders μ, ν in S_r are: firstly that its double locus has the properties given in the previous result, and secondly that Γ has on it at least one $(r-3)$-dimensional variety coresidual to a hyperplane section, which is itself the intersection of two hypersurfaces of S_{r-1}, of orders μ, ν. *P. Du Val* (Bristol).

Rosina, B. A. Sulle superficie algebriche di ordine $2n$ con una conica (almeno) doppia all'infinito (quadriche generalizzate). Ann. Univ. Ferrara. Sez. VII. (N.S.) 2, 141–149 (1953).

A surface F^{2n} with cartesian equation

$$\varphi^n+\varphi^{n-1}f_1+(\text{terms of order }\leqq 2n-2)=0,$$

where φ is a quadratic and f_1 a linear form in the coordinates, cuts the plane at infinity in a conic counted n times, this conic being (at least) double in the surface. It is shown that its diametral properties are in some ways analogous to those of the quadric ($n=1$). A diametral plane being defined [Piazzolla-Beloch, Atti 4° Congresso Un. Mat. Ital., v. 2, Taormina, 1951, Perrella, Roma, 1953, pp. 425–430; these Rev. 15, 249] as the polar plane of a point at infinity (locus of mean centres of the sets traced on the surface by a family of parallel lines) and a principal diametral plane as one which is perpendicular to the corresponding family of lines, the diametral planes of F^{2n} all pass through a point, and in general three of them, perpendicular to each other, are principal; if φ is the product of two linear factors the diametral planes are all parallel to and may all pass through, one line, and two of them, perpendicular to each other, are principal; if φ is a perfect square the diametral planes are all parallel, and one of them is principal; and if $\varphi=kf_1^2$, they all coincide.

These cases are related as in the case of the quadric, to a "discriminant" determinant Δ of order 4, consisting of the discriminant of φ bordered by the coefficients in f_1, the last place being occupied by the constant term of the equation. This last does not enter into any of the arguments, which in fact involve only the first three rows of Δ; and though the surface is called "proper" or "specialised" according as $\Delta\neq 0$ or not, no geometrical characterisation is given of the specialisation involved. *P. Du Val* (Bristol).

Rosina, B. A. Sopra una classe particolare di superficie algebriche (quadriche generalizzate). Atti. Accad. Sci. Ferrara 29, 3–4 (1952).

Brief summary of part of the paper reviewed above, apparently published earlier, but going beyond it in referring to the cases in which the cone $\varphi=0$ is a cone of revolution (when there are a pencil of principal diametral planes, and one other perpendicular to all of them) or is the isotropic cone (when all diametral planes are principal).

P. Du Val (Bristol).

Galafassi, Vittorio Emanuele. In tema di estensione della limitazione di Harnack. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 375–382 (1952).

On associe à un variété algébrique réelle F_k la variété topologique connexe W_{2k}, image des couples symétriques de sa riemannienne, dont les k-cycles diramants correspondent aux nappes réelles, on en déduit aussitôt pour $k=1$ le théorème d'Harnack. Si l'on prend pour modèle de F_k une variété située dans un espace tel que sur la grassmannienne de cet espace, la variété E_{2k} de ses cordes et de ses tangentes soit sans singularités, cette dernière variété définit, si la variété F_k est réelle, une variété U_{2k} topologique fermée; les tangentes à une nappe réelle donnent naissance à un cycle connexe à $2k-1$ dimensions associé à la nappe; de tels cycles séparent les points images de cordes à appui réel de celles à appui imaginaire conjugué, ils forment donc les bords de deux parties de U. On en tire que le nombre de nappes $m\leqq q_{2k-1}+1$, q_{2k-1} est le $(2k-1)$ème rang de connexion de la partie de U, image des cordes à appui réel. Dans le cas des surfaces, en étudiant les relations de cette partie de U_4 et de W_4, on a

$$m\leqq q_3+1=1+q_2-q_1-r_2+2r_1+1-Z,$$

Z ordre de connexion total de la surface, r_1 et r_2 nombres de Betti de W_4.

B. d'Orgeval (Alger).

Galafassi, Vittorio Emanuele. Sulla base reale e sulla connessione delle rigate astratte reali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 260–272 (1952).

An "abstract ruled surface" is one having a pencil (not necessarily rational) of rational curves. If D curves of the pencil are reducible (each to two rational constituents with one intersection), it is known [Severi, Serie, sistemi d'equivalenza e corrispondenze algebriche sulle varietà algebriche, Edizioni Cremonese, Rome, 1942; these Rev. 10, 206] that a base for continuous systems on the surface consists of a unisecant curve A of the pencil, a general member of it L, and one constituent of each of the reducible members. The author shows that if the surface is real, and c' of the degenerate members of the pencil consist of two real constituents, c'' of two conjugate imaginary constituents, and the rest are c_0 pairs of conjugate imaginary members of the pencil (so that $D=c'+c''+2c_0$), a base for real continuous systems on the surface consists of a real unisecant curve (or the sum of two conjugate imaginary unisecants), a general real curve of the pencil, one constituent of each of the c' degenerate curves with real constituents, and c_0 curves, each of which is the sum of two conjugate imaginary curves which are one constituent of each of two conjugate imaginary degenerate members of the pencil.

The total order of connectivity of the real sheets of the surface is $c'-c''+2$, and the number of sheets is $\geqq\frac{1}{2}c''$, every sheet containing either two or none of the real points which are the intersections of conjugate imaginary constituents of degenerate curves of the pencil. *P. Du Val.*

Knight, A. J. A note on overlapped surfaces. J. London Math. Soc. 28, 383–384 (1953).

In Proc. London Math. Soc. (2) 51, 308–324 (1950) [these Rev. 12, 356] D. B. Scott conjectures that all overlapped surfaces are birationally equivalent to ruled surfaces. Knight shows that any surface of irregularity p which contains an irrational pencil of genus p is overlapped.

R. J. Walker (Ithaca, N. Y.).

Knight, A. J. Some surfaces containing irrational pencils of maximum genera. J. London Math. Soc. 29, 38–43 (1954).

The genus of an irrational pencil of curves on an algebraic surface cannot exceed the irregularity of the surface, and a few special cases are known for which the genus equals the irregularity. The author gives a construction for a more general class of such surfaces, for which, in particular, the genus of the pencil and the genus of the curves of the pencil can both be arbitrarily large. *R. J. Walker.*

Knight, A. J. On overlapped algebraic surfaces. J. London Math. Soc. 29, 43–48 (1954).

Complementing an earlier paper [see the second review above] the author shows that any overlapped surface of irregularity p contains a pencil of genus p. The proof is based on a theorem of Severi [Ann. Mat. Pura Appl. (3) 20, 201–215 (1913)] that a surface is mapped into its Picard Variety as a surface or a curve according as it does not or does contain a pencil of maximum genus. *R. J. Walker.*

Morikawa, Hisasi. On abelian varieties. Nagoya Math. J. 6, 151–170 (1953).

En utilisant l'existence de variétés abéliennes quotients [Chow, Proc. Nat. Acad. Sci. U. S. A. 38, 1039–1044 (1952); ces Rev. 14, 580] et le plongement des jacobiennes dans l'espace projectif, l'auteur démontre d'abord que toute variété abélienne abstraite est isomorphe (sur la clôture algébrique de son corps de définition) à une variété abélienne projective; le traitement des phénomènes d'inséparabilité parait insuffisant au rapporteur, mais il pense qu'il est facile de combler cette lacune. Soit alors A une variété abélienne; notons $\sim$ l'équivalence linéaire des diviseurs sur A, X_t le translaté du diviseur X par $t\in A$, et $X\equiv Y$ la relation $X_t-X\sim Y_t-Y$ pour tout $t\in A$ [cf. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948, no. 57; ces Rev. 10, 621]; l'auteur appelle cette dernière l'équivalence algébrique, mais ne démontre pas qu'elle coïncide avec l'équivalence algébrique au sens classique. En notant $G_1(A)$ (resp. $G_a(A)$) le groupe des diviseurs X sur A tels que $X\sim 0$ (resp. $X\equiv 0$), on démontre l'existence d'une variété abélienne A^0 isogène à A [Weil, loc. cit., no. 53] et isomorphe (en tant que groupe additif) à $G_a(A)/G_1(A)$; cette "variété de Picard" de A est déterminée à un isomorphisme purement inséparable près. La méthode des matrices q-adiques de Weil [loc. cit. no. 77] permet de montrer que l'on a $A=A^{00}$ sous certaines conditions peu restrictives.

L'auteur passe alors à une étude arithmétique des anneaux d'endomorphismes de variétés abéliennes. Un lemme montre que, si un corps K de nombres algébriques admet une involution s telle que $\mathrm{Tr}_{K/Q}(x\cdot s(x))>0$ pour tout $x\neq 0$

dans K, alors K est totalement réel ou totalement imaginaire. Il en est donc ainsi du centre de l'anneau des endomorphismes d'une variété abélienne simple.

Le reste du mémoire est consacré à la démonstration purement algébrique du théorème suivant, dû à Frobenius: si A est une variété abélienne sur un corps de caractéristique 0, et si X est un diviseur positif sur A tel que l'on ait $X_t \sim X$ pour seulement un nombre fini de points t de A, alors la dimension $l(X)$ de l'espace vectoriel des fonctions f sur A telles que $(f) \geqq -X$ est égale à la racine carrée de l'entier $\prod_q q^{e(q)}$, où $e(q)$ est l'exposant du nombre premier q dans le déterminant de la matrice q-adique $E_q(X)$ attachée à X [cf. Weil, loc. cit., no. 76]. La démonstration de ce résultat est délicate. Elle fait intervenir tous les résultats précédemment démontrés, une classe de variétés abéliennes (dites "spéciales") qui jouissent de mainte propriété des jacobiennes, et un résultat de Siegel sur la représentation sous forme de sommes de quatre carrés des entiers totalement positifs d'un corps de nombres totalement réel.

P. Samuel (Clermont-Ferrand).

Kawahara, Yûsaku. Remarks on the differential forms of the first kind on algebraic varieties. Nagoya Math. J. 6, 37–40 (1953).

Suppose that ω denotes a differential form on a complete n-dimensional algebraic variety U over the field k. The author proves the following theorems using some simple properties of valuation and specialization rings: (i) If k contains a perfect subfield over which U can be defined and if P is a generic point of U over k, then ω is of the first kind provided its extension $\omega(P)$ to $k(P)$ is of the first kind; and (ii) a differential form ω is of the first kind on a non-singular projective variety U over k if it induces on the generic hyperplane section of dimension $n-1$ (coefficients of the defining equation of the hyperplane are independent indeterminates over k) a differential form of the first kind.

O. F. G. Schilling (Chicago, Ill.).

Matsusaka, Teruhisa. On algebraic families of positive divisors and their associated Varieties on a projective Variety. J. Math. Soc. Japan 5, 113–136 (1953).

Soient V une variété projective non singulière, et C_s le diviseur section de V par une hypersurface de degré s. Au moyen de lemmes montrant que certaines systèmes linéaires de diviseurs sur V sont dépourvus de points base, l'auteur prouve que deux diviseurs positifs algébriquement équivalents ont même genre arithmétique virtuel [cf. Zariski, Ann. of Math. (2) 55, 552–592 (1952); ces Rev. 14, 80]. On dit qu'une famille algébrique maximale (X) de diviseurs positifs ("f.a.m.") est régulière si tout diviseur algébriquement équivalent à 0 est linéairement équivalent à une différence de deux éléments de (X); on dit qu'une f.a.m. est complète si elle est fibrée par l'équivalence linéaire, c.à.d. si tous les systémes linéaires maximaux qu'elle contient ont même dimension. Soit (X) une f.a.m.; il existe un entier m indépendant de (X) (resp. un entier $m'(X)$) tel que pour tout $s \geqq m$ (resp. $s \geqq m'(X)$) l'une des f.a.m. contenant $X + C_s$ soit régulière (resp. régulière et complète); ces résultats utilisent l'existence de la variété de Picard P de V. La variété associée à une f.a.m. régulière est birationnellement équivalente au produit de la variété de Picard P de V par un espace projectif.

P. Samuel (Clermont-Ferrand).

Nakai, Yoshikazu. On the divisors of differential forms on algebraic varieties. J. Math. Soc. Japan 5, 184–199 (1953).

Soient V une variété algébrique sur le domaine universel K, et $F(V)$ le corps des fonctions rationnelles sur V. On appelle espace vectoriel E des formes différentielles de degré 1 sur V le dual de l'espace des K-dérivations de $F(V)$; et on définit les formes différentielles de degré quelconque comme les élements de l'algèbre extérieure de E (sur $F(V)$). On exprime ces formes de la façon habituelle au moyen des dx_i ((x_i) base de transcendance séparante de $F(V)$ sur K, fournie par exemple par un système de paramètres uniformisants en un point simple de V). D'où la notion de corps de définition k ($\subset K$) d'une forme différentielle ω. Si ω est une forme différentielle sur V^r, A une sous variété de dimension $r-1$ de V, (x_i) un système de paramètres uniformisants le long de A, et $\omega = \sum y_{i_1, \ldots, i_p} dx_{i_1} \ldots dx_{i_p}$ l'expression de ω, l'entier $v_A(\omega) = \min (v_A(y_{i_1, \ldots, i_p}))$ ne dépend pas du choix de (x_i); d'où la notion de diviseur de ω, noté (ω). L'auteur montre que toute forme différentielle ω de degré $d \leqq r-1$ sur V^r induit sur une section hyperplane générique W de V une forme $\bar{\omega}$ de même degré, et que l'on a $(\bar{\omega}) = (\omega) \cdot W$ pour $d \leqq r-2$ et $(\bar{\omega}) \geqq (\omega) \cdot W$ pour $d = r-1$. Ceci se généralise aussitôt aux sections linéaires génériques de V de plus basse dimension. D'où, en s'appuyant sur la théorie des courbes, en notant f une fonction sur V et a l'entier $v_A(f)$: $v_A((df)) = a-1$ si $a \not\equiv 0$ (mod. p) et $v_A((df)) \geqq a-1$ si $a \equiv 0$ (mod. p) (p: caractéristique de K).

P. Samuel.

Differential Geometry

Scherk, P. A remark on curves of order n in n-space. Trans. Roy. Soc. Canada. Sect. III. 47, 35–36 (1953).

R_n: real projective n-space. C^n: curve (continuous image of a circle) of order n in R_n. The point s of an arc (continuous image of a closed interval) A is called differentiable if A possesses in s osculating subspaces of every dimension. This note sketches a simple and elegant proof of the following theorem: Every C^n can be uniformly approximated by differentiable C^n's. An auxiliary elliptic metric is introduced in R_n and the given C^n decomposed by means of differentiable points into arbitrarily small subarcs A_i', $i=1, 2, \cdots, p$. Let A_i denote the closed C^n-complement of A_i'. For any fixed i there are $n+1$ combinations of osculating subspaces at the endpoints of A_i' spanning an $(n-1)$-space each; these $(n-1)$-spaces subdivide R_n into 2^n simplexes. One of them, S_i', includes A_i'. The essential step in the construction resting on previous theorems on order-preserving extensions of arcs by I. Sauter and the author, consists in replacing A_i' by a cofinal differentiable arc A_i'' such that $A_i \cup A_i''$ is a curve of order n. This smoothing operation performed for each $i=1, 2, \cdots, p$ yields a differentiable curve D^n of order n. The approximation property follows from the inclusions $A_i'' \subset S_i'$ and the arbitrary smallness of the S_i''s.

Chr. Pauc (Nantes).

Milnor, John. On total curvatures of closed space curves. Math. Scand. 1, 289–296 (1953).

The lengths of the spherical indicatrices of the tangent, binormal and principal normal of a closed space curve C of class 3 are given by $\kappa(C) = \int_C \kappa(s)ds$, $\tau(C) = \int_C |\tau(s)| ds$, and $\omega(C) = \int_C \{\kappa^2(s) + \tau^2(s)\}^{\frac{1}{2}} ds$, respectively, where s denotes arc length, $\kappa(s)$ curvature and $\tau(s)$ torsion. In a previous paper

[Ann. of Math. (2) 52, 248–257 (1950); these Rev. 12, 273] the author studied the total curvature $\kappa(C)$, with special reference to its behaviour under isotopy of space; in the present paper the quantities $\kappa(C)+\tau(C)$ and $\omega(C)$ are studied from the same point of view.

It is shown that, for each isotopy type $\mathfrak{C}$,

$$\varphi(\mathfrak{C})=(2\pi)^{-1}\inf_{C\in\mathfrak{C}}(\kappa(C)+\tau(C))$$

is a positive integer. (It is known [ibid.] that

$$(2\pi)^{-1}\kappa(\mathfrak{C})=(2\pi)^{-1}\inf_{C\in\mathfrak{C}}\kappa(C)$$

is a positive integer; it is not difficult to show that $\inf_{C\in\mathfrak{C}}\tau(C)=0$.) Denoting by $\nu(C_P)$ the number of cusps and inflection points of the projection C_P of C in the xy-plane and by $\mu(C_P)$ the number of points of C_P at which the tangent is parallel to the x-axis, it is shown that $2\varphi(\mathfrak{C})=\min_{C\in\mathfrak{C}}(\mu(C_P)+\nu(C_P))$; this minimum is obtained by a projection C_P that has no cusps. In some simple examples it is even obtained by a projection that has neither cusps nor inflection points, but it is doubtful whether this is always the case. (It is known [ibid.] that $\pi^{-1}\kappa(\mathfrak{C})=\min_{C\in\mathfrak{C}}\mu(C_P)$.)

An upper bound for $\varphi(\mathfrak{C})$ is the least integer $n(\mathfrak{C})$ such that $\mathfrak{C}$ is represented by a closed braid with n strands. It is shown that $\varphi(\mathfrak{C})=n(\mathfrak{C})$ if $\varphi(\mathfrak{C})\leqq 2$. Whether there are any types $\mathfrak{C}$ for which $\varphi(\mathfrak{C})<n(\mathfrak{C})$ seems to be an open question. If a curve C has linking number m with some straight line L (as is the case, for instance, if C is a closed braid with m strands) then $(2\pi)^{-1}(\kappa(C)+\tau(C))\geqq n$. The quantity $(2\pi)^{-1}\inf_{C\in\mathfrak{C}}\omega(C)$, C ranging over representatives of an isotopy type $\mathfrak{C}$ need not be an integer. (For $\mathfrak{C}$ the figure-eight, it lies between 2 and 3.) A question raised by W. Fenchel [Bull. Amer. Math. Soc. 57, 44–54 (1951); these Rev. 12, 634] is answered by the theorem: $(2\pi)^{-1}\sup_C\omega(C)=2$, C ranging over all class 3 curves for which the torsion $\tau(s)$ does not change sign and is not identically zero.

R. H. Fox (Princeton, N. J.).

✓*Villa, M.* **Transformations ponctuelles et transformations crémoniennes.** Colloque de topologie et géométrie différentielle, Strasbourg, 1952, no. 8, 6 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1953.

This paper consists chiefly of a bibliography. The author states: "L'on expose les principaux résultats, obtenus en Italie par différents auteurs pendant les dix dernières années, sur les transformations ponctuelles, qui intéressent de plus près la géométrie algébrique".

Muracchini, Luigi. Sulla geometria differenziale conforme delle trasformazioni puntuali fra due piani. Boll. Un. Mat. Ital. (3) 8, 252–258 (1953).

Conformal properties of point-transformations between two Euclidean planes. Through each point and in each direction there are two elements of the second order E_2 such that there is a circular E_3 containing E_2 to which corresponds a circular E_3 in the other plane. These E_2 belong generally (in each plane) to a system of ∞^2 curves C determined by the trf. (transformation); however their system is only ∞^1 if and only if the trf. is a conformal one; the curves C are undetermined only for circular affinities. The following theorem is proved: A point trf. which transforms ∞^2 circles into circles is necessarily the product of homographic trfs. between the net of lines in one plane and the circles of a net in the other plane.

E. Bompiani.

Upadhyay, M. D. On ϕ-congruences. J. Math. Soc. Japan 5, 95–104 (1953).

L'auteur considère les congruences (dites ϕ-congruences), dont les rayons sont associés à ceux d'une congruence donnée, de façon que ses rayons coupent les rayons homologues de cette dernière congruence sous l'angle constant ϕ. Il cherche l'expression du paramètre de distribution d'une ϕ-congruence, celle de la distance d'un point central quelconque au point correspondant d'une surface de référence particulière introduite par Ranga Chariar, établit les équations des surfaces réglées de la congruence à représentation sphérique minima, et étudie quelques cas particuliers intéressants.

P. Vincensini (Marseille).

Rogovoi, M. R. On Darboux pencils for a nonholonomic surface. Ukrain. Mat. Žurnal 5, 93–98 (1953). (Russian)

In the construction of the pencil of Darboux for a surface, one makes use of the osculating quadric at a point of a surface. Since such quadrics do not exist for a non-holonomic surface, the author constructs by the methods of Bompiani-Klobuchek two pencils of Darboux corresponding to each asymptotic line. For a holonomic surface two pencils then coincide with the pencil of Darboux for the surface.

M. S. Knebelman (Pullman, Wash.).

Rinow, W. An axiomatic foundation of the intrinsic geometry of surfaces. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 3, 227–233 (1953). (Hungarian)

Rinow, W. Über eine axiomatische Begründung der inneren Geometrie der Flächen. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe 3, 1–4 (1954).

Lecture given 17 Dec. 1952 at a celebration of the 150th anniversary of Bolyai's birth.

Sauer, Robert. Differenzengeometrie der infinitesimalen Flächenverbiegung. Monatsh. Math. 57, 177–184 (1953).

Let $\mathbf{x}(u,v)$ be a surface (referred to real asymptotic parameters u, v), $\xi(u,v)$ an arbitrary vector and $\epsilon\to 0$ a constant. Assume that the infinitesimal deformation (1) $\mathbf{x}^*=\mathbf{x}+\epsilon\xi$ is isometric (up to terms ϵ^2). Then (2) $d\mathbf{x}^*=\mathbf{y}\times d\mathbf{x}$ where $\mathbf{y}(u,v)$ is a vector such that (3) $\mathbf{y}_u=\lambda\mathbf{x}_u$, $\mathbf{y}_v=\nu\mathbf{x}_v$ $(\lambda,\nu=0)$ as follows from the integrability conditions of (2). Then it turns out that the parameter of distribution of the ruled surface of the tangents to $u=$const. ($v=$const.) along $v=$const. ($u=$const.) is an invariant of the deformation. The asymptotic lines of $\mathbf{x}$ are deformed in a conjugate net on $\mathbf{y}$. Moreover, the tangent planes of both surfaces are parallel at corresponding points and by virtue of (3) the tangents along $u=$const. on $\mathbf{x}$ are parallel to the $v=$const. tangent lines on $\mathbf{y}$ along $u=$const. Consider now a "surface" whose faces are skew quadrilaterals such that all four edges of any internal vertex are in the same plane. These "surfaces" admit infinitesimal "deformations" (Verknickung) in the sense similar to the one described above. The author discusses this analogy in detail.

V. Hlavatý.

Nitsche, Joachim. Ein mit der Verbiegung der Halbkugel verbundenes Randwertproblem. Arch. Math. 4, 331–336 (1953).

Let S be the boundary of a surface T which is isometric with the half-sphere $x=r\cos\phi$, $y=r\sin\phi$, $z=(1-r^2)^{\frac{1}{2}}$, $0\leqq r\leqq 1$. Let $\rho(\phi)$ represent the curvature of S. Then its derivative $\mu=d\rho/d\phi$ is subject to the following three homo-

geneous integral conditions:

$$\oint \mu d\phi=0, \quad \oint \mu \sin \phi d\phi=0, \quad \oint \mu \cos \phi d\phi=0.$$

The curvature itself must satisfy the further condition

$$\oint \rho d\phi=2\int\int (1-r^2)(1+r^2)^{-1}Hd\sigma,$$

where the double integral is taken over the interior of the unit circle with element of area $d\sigma$, and H represents the mean curvature of T. *A. Douglis* (New York, N. Y.).

Blanuša, Danilo. Les espaces elliptiques plongés isométriquement dans des espaces euclidiens. II. Les espaces elliptiques. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 81–114 (1953). (Serbo-Croatian summary)

[For part I see same Glasnik **8**, 3–23 (1953); these Rev. **14**, 1122.] The problem of embedding an elliptic n-space El_n in R_N so that the geodesics of El_n become circles in R_N, is studied for $n>2$. The derivation of the minimal value $\frac{1}{2}n(n+3)$ for N is followed by theorems on intersections of elliptic spaces and motions in elliptic spaces. The methods used are mainly elementary and synthetic. The paper ends with a remark on embeddings in which geodesics of El_n do not have to become circles in R_N. *A. Nijenhuis.*

Čech, E. Remarks on projective differential geometry. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **3**, 219–225 (1953). (Hungarian)

Lecture given 17 Dec. 1952 at the János Bolyai Mathematics Club's celebration of the 150th anniversary of Bolyai's birth.

Barner, Martin. Zur projektiven Differentialgeometrie der Komplexflächen. II. Konstruktion und integrallose Darstellung spezieller Schiebungen. Math. Ann. **126**, 418–446 (1953).

As a continuation of a previous paper [Math. Ann. **126**, 119–137 (1953); these Rev. **15**, 349], the author studies the special displacements K_2, K_3 and K_4 of a three-dimensional projective space corresponding to the hyperbolic motions in a linear space R_5 of five dimensions under which a hyperplane rotates about a point, a straight line and a plane of dimension two, respectively. One and only one displacement K_4 can be determined by a ruled surface and a linear complex if the generators of the ruled surface are not all in the complex. One and only one displacement K_3 can be determined by a curve and a one-parameter family of complexes if the tangents to the curve are not all in a complex of the family. Moreover, a displacement K_2 is a Clifford displacement. Applications of the theory of displacements are made to obtain explicit representations, free of integration, for complex surfaces of a wider class in which the simplest complex surface was studied by K. Strubecker [Math. Z. **52**, 401–435 (1949); these Rev. **11**, 459]. *C. C. Hsiung.*

Weitzenböck, Roland. Zum Transversalenproblem. I. Geraden im R_3, R_4, und R_5. Monatsh. Math. **57**, 185–198 (1953).

In a projective n-dimensional space, R_n, $(d+1)(n-d)$ given linear R_{n-d-1}'s in general position will have τ_d linear transversal spaces, a linear transversal space being an R_d which has at least one point in common with each of the R_{n-d-1}. It is known that

$$\tau_d=\frac{1!2!3!\cdots d!((d+1)(n-d))!}{n!(n-1)!\cdots(n-d)!}$$

[see van der Waerden, Math. Ann. **113**, 199–205 (1936)]. Thus, given four lines in R_3 in general position, there will be two transversal lines, each one intersecting the given four. After an introduction, the author shows how to find these two transversals from the four given lines, and finds in explicit form the condition that five given lines in R_3 must satisfy if they are to have a transversal. Next the author shows how to find the five transversal planes for six given lines in general position in R_4. The computation is in part a generalization of that of the first case, and it is pointed out how the computations will be further changed in going to 8 lines in R_5 and to the R_n case in general. It is shown how one can obtain the condition under which 7 lines in R_4 will have a transversal, but an explicit form is not given, the difficulty encountered being explained.

In a last section, the concept of associated spaces is presented. Let the ∞^6 lines in R_4 be considered points of the Grassman manifold $\mathfrak{M}_6^5$ in a projective space $\mathfrak{R}_9$. Four given lines in general position in R_4 determine four points on the $\mathfrak{M}_6^5$ and also a linear space $\mathfrak{R}_3$. The $\mathfrak{R}_3$ intersects the $\mathfrak{M}_6^5$ in a fifth point, which represents a fifth line in R_4, the line associated with the given four. This example is generalized, and it is shown that the five transversal planes for six lines in R_4 are a set of associated planes. The 42 transversal planes for 9 given planes in general position in R_5 are a set of 42 associated planes. *A. Schwartz* (New York, N. Y.).

Kovancov, N. I. The canonical pencil as a form of projective symmetry on a surface. Ukrain. Mat. Žurnal **5**, 99–119 (1953). (Russian)

The study of local properties of surfaces is achieved by means of simple geometrical constructions which associate with a point certain other points, lines and curves of second order. These are expressed in terms of the two fundamental projective invariants β and γ and their derivatives. The author observes that these constructs possess symmetry with respect to the asymptotic directions through the point of the surface and it is from this point of view that he obtains the equations of the classical lines, such as the directrices of Wilczynski, the edge of Green, the normal of Fubini, etc. The author also finds a new line whose construction is similar to that of Green. *M. S. Knebelman.*

Ōtsuki, Tominosuke. On some Riemann spaces. Math. J. Okayama Univ. **3**, 65–88 (1953).

The author studies Riemann spaces V_n whose Ricci tensor $K_i{}^j$ satisfies one or both conditions:

(a) $$K_i{}^h K_h{}^j=(n-1)^{-1}K_h{}^h K_i{}^j;$$

(b) $$K^j{}_{i;h}=0,$$

where the semicolon denotes covariant differentiation. Such spaces may be considered as analogues of Einstein spaces. Part I gives an analysis of spaces for which only (a) is satisfied. If the scalar curvature is zero, the space is an Einstein space $(n>2)$ or a locally euclidean space $(n\geqq 2)$. If the scalar curvature is not zero, the Ricci tensor has a null-direction at every point. The space is still of such generality that any Riemann space can be isometrically imbedded in it as a hypersurface. Part II studies spaces for which both (a) and (b) are satisfied. If the scalar curvature is not zero and if the curves whose tangent directions are null directions

of the Ricci tensor are orthogonal trajectories of a family of hypersurfaces, then V_n is a product space of a straight line and an Einstein space with non-zero scalar curvature ($n>3$) or a surface of non-zero constant curvature. The converse of this result is also true. *S. Chern* (Chicago, Ill.).

Gustin, Wm. Nonexistence of conformal singularities in solid spaces. J. Rational Mech. Anal. **3**, 73–76 (1954).

In a conformal mapping of X into Y, where X and Y are Riemannian spaces of dimension $n>2$, a conformal singularity is a point of X at which the conformal factor vanishes. It is shown that if the metrics of X and Y are positive definite, and if a conformal mapping $X\rightarrow Y$ is analytic and not constant, then X has no conformal singularities. It is remarked that this theorem does not extend to spaces with indefinite metrics. *A. G. Walker* (Liverpool).

Berger, Marcel. Sur les groupes d'holonomie des variétés riemanniennes non symétriques. C. R. Acad. Sci. Paris **237**, 1306–1308 (1953).

The author gives an enumeration of the groups (up to an isomorphism) that could occur as the restricted homogeneous holonomy groups of non locally reducible, non-symmetric Riemannian spaces. He finds for V_{2n+1}: $SO(2n+1)$; for V_{4n+2}: $SO(4n+2)$, $SU(2n+1)$, $U(2n+1)$; for V_{4n}: $SO(4n)$, $SU(2n)$, $U(2n)$, $Sp(n)$, $T^1\otimes Sp(n)$, $SU(2)\otimes Sp(n)$. Besides, V_8 admits $SO(7)$, and V_{16} admits $SO(9)$.
A. Nijenhuis (Princeton, N. J.).

Nomizu, Katsumi. Sur les transformations affines d'une variété riemannienne. C. R. Acad. Sci. Paris **237**, 1308–1310 (1953).

A number of relations are established between the groups of "affine motions" (A) and the group of isometries (I) in a Riemannian space V. The results are summarised in his Theorem II: If the Euclidean part of a Riemannian space V is of dimension $\leqq 1$, then (1) every compact subgroup of the identity component $A_0(V)$ is contained in $I(V)$; (2) every connected semi-simple Lie subgroup of $A_0(V)$ is contained in $I_0(V)$; (3) the derived group $[A_0(V), A_0(V)]$ is contained in $I_0(V)$. *A. Nijenhuis* (Princeton, N. J.).

Nomizu, Katsumi. Application de l'étude des transformations affines aux espaces homogènes riemanniens. C. R. Acad. Sci. Paris **237**, 1386–1387 (1953).

A homogeneous Riemannian space is a coset space G/H (H compact) with a metric that is invariant under G. The first problem investigated is the reducibility of G/H. If G is simply connected, every factor is a homogeneous space itself. If G is simple and effective on G/H, then G/H is irreducible and non-Euclidean. The second problem deals with covariant constant fields: if G is simple and has no center, every covariant constant form on G/H is invariant under G. *A. Nijenhuis* (Princeton, N. J.).

Nomizu, Katsumi. On the group of affine transformations of an affinely connected manifold. Proc. Amer. Math. Soc. **4**, 816–823 (1953).

The author proves that the group $A(M)$ of all affine transformations of a complete affinely connected manifold is a Lie group. This partially generalizes the theorem of the reviewer and Steenrod which states that the group of all isometries of a Riemannian manifold is a Lie group [Ann. of Math. (2) **40**, 400–416 (1939)]. In the Riemannian case no completeness assumption was needed; the author leaves open the question as to whether the completeness assumption is really needed in the affine case. The method of proof consists of showing that $A(M)$ is locally compact under the compact-open topology, and that any member of $A(M)$ leaving a non-empty open set of M pointwise fixed is the identity; it then follows from a theorem of Bochner and Montgomery that $A(M)$ is a Lie group [Ann. of Math. (2) **47**, 639–653 (1946); these Rev. **8**, 253]. *S. B. Myers.*

Nomizu, Katsumi. Invariant affine connections on homogeneous spaces. Amer. J. Math. **76**, 33–65 (1954).

The paper clarifies and generalizes the work of E. Cartan on the relationship between the theory of connections and the geometry of Lie groups and their homogeneous spaces. More specifically, it is concerned with the study of affine connections which can be defined on a homogeneous space G/H (G a connected Lie group and H a closed subgroup) and which remain invariant under the action of G. Chapter I gives a new definition, due to J. L. Koszul, of an affine connection. From this definition the usual concepts in affine connections, such as torsion and curvature tensors, covariant differentiation, parallelism, and holonomy groups, are developed, the last without details. Chapter II studies invariant affine connections on reductive homogeneous spaces. G/H is called reductive, if in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ (direct sum of vector spaces) and $\mathrm{ad}(h)\mathfrak{m}\subset\mathfrak{m}$ for all $h\in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_0 of H and $\mathrm{ad}(h)$ denotes the adjoint representation of H in $\mathfrak{g}$. A homogeneous space is reductive, if H is compact or is connected and semi-simple or is discrete. A fundamental existence theorem asserts that there is a one-one correspondence between the set of all invariant affine connections on G/H and the set of all bilinear functions α on $\mathfrak{m}\times\mathfrak{m}$ with values in $\mathfrak{m}$ which are invariant under $\mathrm{ad}(H)$. In terms of α, the torsion tensor, the curvature tensor, and the covariant differential of the curvature tensor are computed. Among these invariant affine connections there are two, to be called respectively the canonical affine connections of the first and second kind, which include the connections defined by E. Cartan in the space of a connected Lie group. The structure of the Lie algebra of the restricted holonomy group of the canonical affine connection of the second kind is described. Chapters III and IV study symmetric spaces. Various theorems are established, generalizing known theorems of E. Cartan on symmetric Riemannian spaces. In particular, it is notable that, unlike the Riemannian case, the condition that an affine connection be invariant under parallelism is weaker than that it be symmetric. *S. Chern.*

Kawaguchi, Akitsugu. On a higher order space with the connection belonging to a Lie group. Rend. Circ. Mat. Palermo (2) **1** (1952), 361–372 (1953).

Die metrischen und nicht-metrischen Punkträume mit linearer Übertragung, lassen sich im wesentlichen dadurch charakterisieren, dass benachbarte Tangentialräume eine Abbildung aufeinander gestatten, die eine Kongruenz, Affinität, oder Projektivität ist. Damit sind diese Geometrien in gewissem Sinne dem Kleinschen Erlanger-Programm unterordnet. Verf. zeigt in vorliegender Abhandlung, dass sich Kawaguchische Räume (Mannigfaltigkeiten von Flächenelementen höherer Dimension mit Übertragung) konstruieren lassen, deren Zusammenhang (invariantes Differential), durch eine beliebige Liesche Gruppe bestimmt ist. Nachdem im §1 die notwendigen Ergebnisse über erweiterte Liesche Transformationsgruppen zusam-

mengestellt werden, erhält der §2 diejenigen vier Postulate, vermöge denen eine durch die Gruppe $\mathfrak{G}$ bestimmte Übertragung konstruiert werden kann. Das I-te Postulat fordert, dass zu jedem Flächenelement höherer Ordnung ein Tangentialraum gehört, deren Geometrie durch die entsprechende Erweiterung der Gruppe $\mathfrak{G}$ bestimmt ist. Durch das II-te Postulat werden die Elemente benachbarter Tangentialräume ein-eindeutig durch eine zu $\mathfrak{G}$ gehörige Abbildung aufeinander bezogen. Postulat III fordert die Existenz einer differenzierbaren Abbildung der Flächenelemente höherer Ordnung eines Punktes auf Pseudo-Exvektoren dieses Punktes. Das IV-te Postulat ordnet einem jeden Flächenelement höherer Ordnung der Mannigfaltigkeit ein Element im Tangentialraum zu. Die ersten beiden Postulate ermöglichen bereits die Einführung eines ausschliesslich von $\mathfrak{G}$ abhängenden invarianten Differentials. Verf. zeigt, dass für lineare Tangentialräume das invariante Differential ein linearer Operator ist. Ferner beweist er, dass die Differentialinvarianten der Mannigfaltigkeit, eine infinitesimale Parallelübertragung gestatten. In §3 wird auf Grund der beiden letzten Postulate eine Basisübertragung für die Flächenelemente höherer Ordnung konstruiert. Der 4-te Paragraph ist der Krümmungs- und Äquivalenztheorie gewidmet. Die Integrabilitätsbedingungen der das invariante Differential bestimmenden Pfaffschen Formen führen zu den Krümmungsgrössen der Mannigfaltigkeit. Falls zwei, auf die obige Weise konstruierte Mannigfaltigkeiten durch eine Koordinatentransformation in einander übergehen, dann müssen auch die das invariante Differential bestimmenden Pfaffschen Formen und die sämtlichen äusseren Ableitungen derselben bei dieser Transformation in einander übergehen. Umgekehrt bestimmen die genannten Gleichungen ein System, dessen Verträglichkeit die Äquivalenz der Mannigfaltigkeiten nach sich zieht. *O. Varga* (Debrecen).

Kurita, Minoru. On the vector in homogeneous spaces. Nagoya Math. J. **5**, 1–33 (1953).

Es sei $\mathfrak{H}$ eine Liesche Transformationsgruppe und $\mathfrak{L}$ diejenige Untergruppe, die einen gewissen Punkt des Raumes fest lässt. Der homogene Raum ist dann durch die Faktorgruppe $\mathfrak{H}/\mathfrak{L}$ bestimmt. Die durch die Pfaffschen Formen $w_1, \cdots, w_n, \cdots, w_r$ bestimmten Relativkomponenten von $\mathfrak{H}$ seien gleich so gewählt, dass die ersten w_i $(i=1, \cdots, n)$ die prinzipalen Relativkomponenten des homogenen Raumes sind, während die w_α $(\alpha=n+1, \cdots, r)$ sekundäre Relativkomponenten bedeuten. Die Vektoren in dem Tangentialraum eines Punktes besitzen eine Transformationsgesetz, das durch die, zu einer Transformation von $\mathfrak{L}$ gehörige, lineare adjungierte Gruppe bestimmt ist. Es folgt daraus, dass auch die w_i kontravariante Komponenten eines infinitesimalen Vektors sind. Sind c_{KMN} $(K, M, N, =1, \cdots, n, \cdots, r)$ die Strukturkonstanten, dann wird das kovariante Differential eines Vektors v_i durch

$$Dv_i = dv_i - \sum_{\alpha, k} c_{\alpha k i} w_\alpha v_k$$

bestimmt. Verf. beweist, dass Dv_i dann und nur dann ein Vektor ist, falls $c_{\alpha i \beta}=0$ gilt. Der Satz war schon von E. Cartan [Ann. of Math. (2) **38**, 1–13 (1937)] angegeben, der Beweis aber bloss skizziert. Die Bildung des kovarianten Differentials ist nicht eindeutig, da es von einer bestimmten Wahl der w_i abhängt. Verf. gibt notwendige und hinreichende Bedingungen für die Eindeutigkeit dieser Operation an. Bei Beschränkung auf Räume in denen das kovariante Differential eines Vektors wieder zu Vektoren führt, untersucht Verf. diejenigen Räume, die einen vom Weg unabhängigen sogenannten absoluten Parallelismus zulassen. Als notwendig und hinreichend findet Verf., dass $\mathfrak{H}/\mathfrak{L}$ durch $\mathfrak{L}$ und eine n-parametrige invariante Untergruppe erzeugt wird, die einfach transitiv auf die Punkte des Raumes wird. Falls der Raum symmetrisch ist, dann folgt aus der Forderung des absoluten Parallelismus, dass der Raum ein affiner Raum ist. Seine Fundamentalgruppe enthält die Gruppe aller Translationen, während die Drehungsgruppe nicht die ganze lineare Gruppe sein muss. Verf. zeigt, dass die Geodätischen des Raumes, die als autoparallele Kurven definiert sind, durch eine einparametrige Untergruppe der gegebenen Gruppe erzeugt werden. Weiter wird gezeigt, dass die Geodätischen die extremalen der Integralinvarianten $\int f(w)$ sind. Dabei ist $f(w)$ eine solche Invariante von $\mathfrak{H}/\mathfrak{L}$ die linear und homogen in den $w_1, \cdots, w_n$ ist, und den Relationen $\sum_{j,i} c_{jki} w_j \partial f/\partial w_i=0$ genügt. Falls der Raum eine Riemannsche Metrik gestattet, die invariant hinsichtlich der gegebene Gruppe ist, dann gibt es zwei Möglichkeiten für die Bildung des kovarianten Differentials und daher zwei Arten von Parallelübertragungen. Verf. zeigt, dass die beiden Parallelübertragungen für einen homogenen Raum dann und nur dann zusammenfallen, falls derselbe symmetrisch ist. Schliesslich behandelt Verf. allgemeine Räume, die einen, durch Pfaffsche Formen gegebenen, Zusammenhang besitzen. Die Tangentialräume dieser Räume lassen sich längs einer Kurve auf einen homogenen Raum abwickeln. Nach der Definition des Krümmungstensors und Torsionsvektors zeigt Verf., dass die betrachteten Räume homogen sind falls der Krümmungstensor und der Torsionsvektor verschwindet. *O. Varga.*

Kurita, Minoru. Grouptheoretical characterization of projective space and conformal space. Nagoya Math. J. **5**, 59–74 (1953).

Diese Arbeit schliesst sich eng an die vorstehend referierte an, sodass wir uns im folgenden auch auf dieses Referat beziehen werden. Es werden hier zwei konkrete Gruppen, nämlich die projektive und konforme untersucht. Geht man von der, diesem Falle entsprechenden Gruppe $\mathfrak{L}$ aus, so liefert das kovariante Differential eines Vektors keinen Vektor. Verf. kann aber eine gewisse Untergruppe der vollen linearen Gruppe angeben für die dies zutrifft. Eine gleiche Feststellung gilt für beliebige homogene Räume. Verf. bezeichnet einen homogenen Raum aus einem solchen von projektiven Typus, falls die Fundamentalgruppe $\mathfrak{H}$ derselben durch die gesamte Drehungsgruppe und eine solche kommutative Untergruppe von $\mathfrak{H}$ erzeugt wird, die einfach-transitiv auf die Punkte des Raumes wirkt. Die projektiven und konformen Räume sind stets von projektiven Typus und lassen sich unter diesen durch gewisse Bedingungen charakterisieren. *O. Varga* (Debrecen).

Vrănceanu, G. Sur les espaces à connexion projective dont le groupe d'holonomie fixe une quadrique. Tôhoku Math. J. (2) **4**, 103–108 (1952).

Es sei P_n ein projektiv zusammenhängender Raum mit normaler Übertragung. Die Hyperfläche Q_{n-1} im Tangentialraum eines Punktes P enthalte diesen Punkt nicht. Sasaki und Yano bewiesen, dass unter diesen Voraussetzungen der Raum P_n konform zu einem Einsteinschen Raum konstanter skalarer Krümmung ist, falls die Holonomiegruppe von P_n die Hyperfläche Q_{n-1} fest lässt. Verf. gibt einen überraschend einfachen Beweis dieses Satzes unter Benützung von orthogonalen Kongruenzen. Verf. behandelt auch den Fall indem die Q_{n-1} den Punkt P enthält und

zeigt, dass sich dann dem P_n ein anholonomes Gebilde V_n^{n-1} zuordnen lässt, das konform zu einem Raum C_{n-1} mit konformen Zusammenhang ist. Mit Hilfe dieser Resultate kann Verf. die Sasakischen Untersuchungen die sich auch solche Räume mit normalen konformen Zusammenhang beziehen, deren Holonomiegruppe eine Hyperkugel fest lassen, auf einfache Weise herleiten. *O. Varga.*

HAVE

***Duschek, Adalbert, und Hochrainer, August. Grundzüge der Tensorrechnung in analytischer Darstellung. II. Teil: Tensoranalysis.** Springer-Verlag, Wien, 1950. viii+338 pp. $6.00.

In the second part of their introduction to tensor analysis, the authors furnish a stimulating discussion of the applications of vector and tensor analysis to differential geometry, potential theory, and Riemannian geometry. The first and third topics (sections 16 to 22 and 33 to 38) are due to Duschek, and the applications to potential theory (sections 23 to 32) are, in the main, the work of Hochrainer.

The approach to differential geometry is by use of vector analysis and covers the elements of the classical theory. In curve theory, the curvature, the torsion, the osculating plane, etc., and the Frenet formulas are discussed. Surface theory treats the first and second fundamental forms of a surface in three-dimensional Euclidean space. The properties of asymptotic, conjugate, and principal directions are studied. Developable surfaces are discussed by means of the Weingarten formulas.

The potential theory discussed in the text deals with the scalar fields associated with Laplace's or Poisson's equations. In addition, the vector fields which possess a scalar or vector potential are examined. As is customary, the theorems of Stokes, Gauss, and Green are developed. Then, the point source (or the elementary solution) and the dipole solutions of Laplace's equation are discussed. This leads to a brief treatment of Green's functions, boundary-value problems, and Poisson's equation. The decomposition of the divergence of a general field in terms of its associated principal normal, binormal and their curvatures are determined. In addition, vector fields normal to ∞^1 surfaces are studied. Finally, plane vector and tensor fields are discussed in some detail. This includes a study of the relation between vectors and complex numbers, Green's theorem, and plane potential theory.

The section on Riemannian geometry furnishes an introduction to this subject. The authors have included material on parallel displacement, the curvature tensor, and surface theory (two-dimensional surfaces in three-dimensional Euclidean space). In this last topic, the second fundamental tensor is introduced, the Gauss and Mainardi-Codazzi relations are found, and a detailed proof of the fundamental theorem of surface theory is given. Since most texts do not furnish such a proof, this section should be of interest and value to the student (and even to the more advanced reader who has forgotten the classical theory of a mixed system of first order partial differential equations). The text proper concludes by studying some special coordinate systems. Finally, the solutions of the problems in part 1 and the present part 2 are given.

The presentation is lucid, and a fairly wide range of topics have been treated. The reviewer would have liked a discussion of the "physical components" of a tensor. This topic may be included in the forthcoming part 3. Generally speaking, the level of the discussion is that which is appropriate for the more mature student. The authors have written an interesting and valuable text. *N. Coburn.*

NUMERICAL AND GRAPHICAL METHODS

HAVE

***Householder, Alston S. Principles of numerical analysis.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. x+274 pp. $6.00.

The purpose of this book is to present numerical procedures suitable for use in an automatic sequence digital calculator. The following topics are treated in order: Errors in computation, the solution of linear systems, the solution of polynomial equations, proper values and vectors of matrices, polynomial interpolation and approximations, numerical integration and differentiation and a brief account of the Monte Carlo method. The book is distinguished by a sound theoretical approach so that the intellectual content is quite satisfactory. Thus the treatment of linear problems is associated with a reasonably complete discussion of finite-dimensional vector spaces and the more general algebraic problem with polynomial elimination. Because of its logical development and range of mathematical topics, the book would be a satisfactory textbook for an upper college or first year graduate course in numerical methods. The reviewer would prefer a more explicit tie-in with probability concepts in the discussion of errors and a more detailed discussion of stability in computation, but these comments probably reflect spatial limitations. *F. J. Murray.*

***Zurmühl, R. Praktische Mathematik für Ingenieure und Physiker.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953. x+481 pp. DM 28.50.

This book is intended to supplement the theoretical training given in the usual calculus and elementary differential equation courses for engineers. The author achieves this aim by presenting a complete account of numerical methods that may be used in the every day application of mathematics to a wide variety of problems in engineering and physical sciences. The text is replete with numerical examples and illustrative figures. The derivation of most formulae from first principles and an evaluation of their usefulness is given. Of course, the author recommends the use of desk calculators in carrying out the numerical work. A partial list of the topics that are treated follows.

Chapter I is an eight page introduction to calculations with complex numbers with desk calculators and with the slide rule. Chapter II, on roots of equations, gives the standard graphical and iterative techniques (including Newton's method), used to determine the zeros of nonlinear functions. In particular, a treatment is given of Horner's method, Graeffe's method, and the technique of Brodetsky and Smeal for finding the complex roots of polynomials. A presentation is made of the Routh and Hurwitz criteria for determining the number of complex roots in the left half-plane (without the use of the Cauchy integral formula). Chapter III treats the techniques for solving a system of linear algebraic equations by the Gauss elimination method and the Gauss-Seidel iteration process. Hessenberg's method for determining the characteristic equation and the characteristic vectors of a matrix is given. An explanation is made of iterative techniques for solving the characteristic value problem.

Chapter IV treats the Lagrange interpolation polynomial

in its various forms, the Hermite "interpolation" polynomial and various convenient finite difference formulae (including divided differences). In addition, some techniques for graphical evaluation of single and iterated integrals are given. Chapter V treats smoothing techniques and least squares fitting of observations. A presentation of the fundamental methods of probability and statistics to estimate the reliability of direct and indirect measurements is given. Chapter VI discusses the approximation of functions by expansions in Fourier series (harmonic analysis), Legendre and Tschebyscheff polynomials, etc. A discussion of the smoothing of data by differencing is given.

Chapter VII is on the numerical integration of the initial-value problem for ordinary differential equations. A large variety of methods from the simple Euler-Cauchy polygonal approximation to sophisticated variations of the Runge-Kutta technique is given. Chapter VIII, the longest, treats the boundary-value and eigenvalue problems for ordinary differential equations. The author gives a general theoretical discussion of the subject together with a treatment of the finite difference, collocation, Rayleigh-Ritz, numerical iteration, etc., techniques for numerical solution. The author is to be complimented on having carefully presented such a wide range of worthwhile subjects. *E. Isaacson.*

David, F. N., and Kendall, M. G. Tables of symmetric functions. IV. Biometrika **40**, 427–446 (1953).

The tables of this paper are a continuation of series of fundamental tables of symmetric functions for weights not exceeding twelve. Four varieties of symmetric functions of x_i are treated, namely: (M) monomial, such as $(211)=\sum x_i^2 x_j x_k$; (U) unitary, or elementary, such as $a_3=\sum x_i x_j x_k$; (S) power sums $s_k=\sum x_i^k$; (H) homogeneous product-sums h_r generated by the identity

$$\sum_{r=0}^{\infty} h_r t^r = \prod_i (1-x_i t)^{-1}.$$

The tables give the coefficients in the linear combinations relating one variety with another. Each table has its inverse. The present table may be described as the MH-HM table. From it one may read, for example, that

$$h_1 h_2 = \sum x_i^3 + 2\sum x_i^2 x_j + 3\sum x_i x_j x_k$$

and

$$\sum x_i^2 x_j = -2k_1^3 + 5h_1 h_2 - 3h_3.$$

Parts I, II and III which have already appeared [Biometrika **36**, 431–449 (1949); **38**, 435–462 (1951); these Rev. **11**, 488; **13**, 781] are the MS-SM, MU-UM and UH-HU tables respectively. Part V, the US-SU tables, is promised. There will be no part VI since the HS-SH table is essentially the same as US-SU except for signs. There is a very brief introduction giving one problem to which the table may be applied. *D. H. Lehmer* (Berkeley, Calif.).

Emersleben, O. Anwendungen zahlentheoretischer Abschätzungen bei numerischen Rechnungen. Z. Angew. Math. Mech. **33**, 265–268 (1953).

The author discusses the role played by the number $r_p(n)$ of representations of n as the sum of p integral squares in the determination of certain numerical constants occurring in crystallography which are special values of the Epstein zeta-function. Instances are given in which such series are (a) evaluated in closed form, (b) proved divergent, and (c) computed numerically with considerable accuracy, using arithmetic properties of $\mu_p(n)$ and its generating function. *D. H. Lehmer* (Berkeley, Calif.).

Baeschlin, C. F. Die Berechnung des Logarithmus einer Primzahl. Schweizer. Z. Vermess. Kulturtech. Photogram. **52**, 8–11 (1954).

Stiefel, E. Zur Interpolation von tabellierten Funktionen durch Exponentialsummen und zur Berechnung von Eigenwerten aus den Schwarzschen Konstanten. Z. Angew. Math. Mech. **33**, 260–262 (1953).

The paper describes a difference-quotient scheme, developed by H. Rutishauser, and defined by the relations (using essentially a forward difference notation):

$$Q_i^k = f_{i+1}^k / f_i^k; \quad \Delta_i^k = Q_{i+1}^k - Q_i^k;$$
$$\bar{\Delta}_i^k = \Delta_i^k + \Delta_{i+1}^{k-1}; \quad f_i^{k+1} = f_{i+1}^k \cdot \bar{\Delta}_i^k.$$

If $f(x)$ is expressible as the sum of n exponential terms of the form $a_r \lambda_r^x$, then the nth difference, calculated by the above scheme is zero. The use of these differences is indicated to extrapolate in a given function, and to the calculation of latent roots and vectors of a square matrix. *D. C. Gilles* (London).

Wishart, John, and Metakides, Theocharis. Orthogonal polynomial fitting. Biometrika **40**, 361–369 (1953).

A detailed computing procedure is given for weighted multiple regression analysis, with an application to the problem of orthogonal polynomial fitting when the observations have unequal weights or are not equally spaced. The procedure given is often called the abbreviated Doolittle method; computing details have been described elsewhere by many authors, including this reviewer. The present computing procedure includes an added feature for polynomial fitting: the appropriate fitted curve for successive degrees of the polynomial. *R. L. Anderson.*

Chapanis, A. Notes on an approximation method for fitting parabolic equations to experimental data. Psychometrika **18**, 327–336 (1953).

Morris, J., and Head, J. W. Note on Lin's iteration process for the extraction of complex roots of algebraic equations. Quart. J. Mech. Appl. Math. **6**, 391–397 (1953).

S. N. Lin [J. Math. Physics **20**, 231–242 (1941); these Rev. **3**, 153] gave a method for the determination of quadratic factors of algebraic polynomials. In this note convergence criteria are given for Lin's process when a quadratic factor is extracted from a quartic. The way in which a root is reached when a linear factor is extracted by the process is examined for an equation of general degree whose factors are known. *E. Frank* (Chicago, Ill.).

Purcell, Everett W. The vector method of solving simultaneous linear equations. J. Math. Physics **32**, 180–183 (1953).

In this article [submitted in December 1952] the author gives the formulas for a promising new direct method for solving a system of linear algebraic equations. The reviewer will add a geometrical interpretation embodying Motzkin's independent proposal of the method [summer of 1953, unpublished]. The reviewer's A_i is the author's V_i.

Let points X in n-dimensional projective space have homogeneous coordinates $x_1, \cdots, x_{n+1}$. For $i=1, \cdots, n$, let the ith given equation, $A_i \cdot X \equiv \sum_{j=1}^{n+1} a_{ij} x_j = 0$, be interpreted as requiring that X lies on the hyperplane $\mathfrak{A}_i$. The author picks $n+1$ points $V_1^0=(1,0,\cdots,0), \cdots, V_{n+1}^0=(0,0,\cdots,1)$. Using V_1^0 as a center of projection, project the other points onto $\mathfrak{A}_1$, to obtain the n points $V_1^1, \cdots, V_n^1$. Using V_1^1 as

a center of projection, project the other points (within $\mathfrak{A}_1$) onto $\mathfrak{A}_2$, to obtain the $n-1$ points $V_1^2, \cdots, V_{n-1}^2$. Continuing in this way, the point V_1^n is within $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$, and thus solves the system.

Each projection is simple computationally:

$$V_i^j = C_i^j V_1^{j-1} + V_{i+1}^{j-1},$$

where $C_i^j = -(A_j \cdot V_{i+1}^{j-1})/(A_j \cdot V_1^{j-1})$. The determinant $|a_{ij}|$ $(i, j=1, \cdots, n)$ is equal to $\prod_{i=1}^n (A_i \cdot V_1^{i-1})$. The author's method requires the same number of operations as elimination, but uses less storage space (approximately $n^2/4+n+2$ cells for computed results), and is therefore recommended for such equipment as the IBM Card-Programmed Calculator. [It also has the advantage of never altering the given a_{ij}.] A numerical example is given with $n=4$.

G. E. Forsythe (Los Angeles, Calif.).

Orloff, Constantin. Méthode spectrale pratique d'évaluation numérique des déterminants et de résolution du système d'équations linéaires. Bull. Soc. Math. Phys. Serbie 5, no. 1–2, 17–30 (1953). (Serbo-Croatian summary)

An ingenious method is presented for using desk calculators to obtain scalar products and linear combinations of vectors of low order whose components are small integers. A vector $a=\{a_1, \cdots, a_n\}$ is represented on a calculator by the integer $S_a = a_1 10^{nh} + a_2 10^{(n-1)h} + \cdots + a_n$, called the "spectrum" of a with "rhythm" h [K. Orloff, Acad. Serbe. Bull. Acad. Sci. Math. Nat. A. 1, 63–65, 91–96 (1933)]. One must have $10^h/2 > \alpha = \max |a_i|$. The "inverse spectrum" Σ_a is the integer $a_n 10^{nh} + \cdots + a_2 10^h + a_1$.

If $\beta = \max |b_i|$, and $10^h > 2n\alpha\beta$, then the scalar product $a \cdot b = \sum a_i b_i$ is the middle h digits of the arithmetic product $S_a \Sigma_b$. For example, if $a=(2, 3, 1, 2)$, $b=(4, 2, -1, 3)$, $a \cdot b = 19$, one has $h=2$, $S_a = 2030102$, $\Sigma_b = 2990204$, and $S_a \Sigma_b = 06070419120808$. In a more obvious way one can form linear combinations like $3a+4b$ by computing $4S_a+3S_b$. It is sometimes necessary, however, to increase h before such operations. The spectra may be used to evaluate small determinants of small integers by successive condensation (numerical example given for $n=4$), and to solve small linear systems in various ways. *G. E. Forsythe.*

***Householder, Alston S. The geometry of some iterative methods of solving linear systems.** Simultaneous linear equations and the determination of eigenvalues, pp. 35–37. National Bureau of Standards Applied Mathematics Series, No. 29, U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

Some standard infinite iterative methods for solving systems are given a unified geometric interpretation in terms of general projection methods. A positive definite matrix A is interpreted as a metric in n-dimensional space. Solving the linear system $Ax=y$ is interpreted as finding the contravariant representation x of a vector whose covariant representation is y. Given rectangular matrices $U_0, U_1, \cdots, U_\alpha, \cdots$ of contravariant column vectors, the author defines the projection operators $P_\alpha = U_\alpha (U_\alpha^T A U_\alpha)^{-1} U_\alpha^T$. For a given approximation x_α to x, let $y_\alpha = Ax_\alpha$ and $r_\alpha = y - y_\alpha$. The next approximation to x is defined to be $x_{\alpha+1} = x_\alpha + P_\alpha r_\alpha$. To make it easy to compute the inverse matrix $(U_\alpha^T A U_\alpha)^{-1}$, in most processes one takes U_α to be a single vector. E.g., in the method of steepest descents $U_\alpha = r_\alpha$, while in the Seidel method the U_α are the unit vectors in some order. But use of several columns in U_α is sometimes advantageous.

Two geometrical interpretations of a linear system are discussed when A is nonsymmetric. In the first, one seeks the point x common to a system of hyperplanes $Ax=y$. In the other, one seeks to find multipliers (elements of x) so that y is resolved along the column vectors of A. Projection methods are described for each interpretation. Inconsistent systems are also discussed.

The author's exposition was stimulated by an unsigned set of notes distributed by the National Bureau of Standards Institute for Numerical Analysis in 1949 [known to the reviewer to have been written by J. B. Rosser in collaboration with M. R. Hestenes]. *G. E. Forsythe.*

Healy, M. J. R., and Dyke, G. V. A Hollerith technique for the solution of normal equations. J. Amer. Statist. Assoc. 48, 809–815 (1953).

***Hestenes, Magnus R. Determination of eigenvalues and eigenvectors of matrices.** Simultaneous linear equations and the determination of eigenvalues, pp. 89–94. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

This paper is a review of several methods for obtaining eigenvalues and eigenvectors of finite matrices studied at the National Bureau of Standards Institute for Numerical Analysis from 1948 to 1951. The methods discussed are: (1) The "power method" of forming $A^p x$, for large p; (2) the gradient method for getting an eigenvalue, for symmetric A [Hestenes and Karush, J. Research Nat. Bur. Standards 47, 45–61 (1951); these Rev. 13, 283]; (3) a generalization of (2) involving minimization in more dimensions [Karush, Pacific J. Math. 1, 233–248 (1951); these Rev. 13, 388]; (4) a finite iteration for getting the characteristic polynomial of A [Lanczos, J. Research Nat. Bur. Standards 45, 255–282 (1950); these Rev. 13, 163], and an extension with a more general metric, valid for symmetric A, presented in the present paper; (5) alternating application of the gradient method on two complementary subspaces, useful when the whole matrix is too large to compute with; (6) proposed gradient methods for treating problems $Ax=\lambda Bx$, for arbitrary A, B, based on the treatment for Hermitian A and (definite) B by M. R. Hestenes and W. Karush [J. Research Nat. Bur. Standards 47, 471–478 (1951); these Rev. 14, 236]. One proposal in (6) is to try to minimize a generalized length $|A-\sigma B|_H$ as a function of a complex variable σ. No conclusion is reached as to its computational feasibility. *G. E. Forsythe.*

***Givens, Wallace. A method of computing eigenvalues and eigenvectors suggested by classical results on symmetric matrices.** Simultaneous linear equations and the determination of eigenvalues, pp. 117–122. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

By a sequence of at most $\frac{1}{2}(n-1)(n-2)$ rotations, each in a plane spanned by two coordinate vectors, a real symmetric matrix A can be reduced to a "triple diagonal" symmetric matrix S determined by

$$s_{ij} = \begin{cases} \alpha_i & (j=i), \\ \beta_i & (j=i+1), \\ 0 & (j \geq i+2). \end{cases}$$

This $S=T'AT$, where T is the orthogonal matrix built up from the rotations. The procedure is a terminating al-

gorithm making new use of the rotations employed in the infinite iterative procedure for diagonalizing A recommended by von Neumann and Goldstine [and by Jacobi, J. Reine Angew. Math. **30**, 51–94 (1846)].

The successive principal minors $f_i(\lambda)$ of $\lambda I-S$ can be computed from the recursion $f_i(\lambda)=(\lambda-\alpha_i)f_{i-1}(\lambda)-\beta_{i-1}^2f_{i-2}(\lambda)$, where $f_{-1}=0$, $f_0=1$, and $i=1, 2, \cdots, n$. If no $\beta_i=0$, the functions $f_n(\lambda), f_{n-1}(\lambda), \cdots, f_0(\lambda)$ form a Sturm sequence, and can thus be used to compute the roots of f_n (the eigenvalues of A). If some $\beta_j=0$, then S decomposes into a direct sum of symmetric matrices. In any case the columns of T are the eigenvectors of A. The method is believed to be well suited to automatic digital calculation; a note added in proof (1952) confirms its stability.

The author mentions that the triple diagonal form (obtained otherwise) and the same recursion relations have been discussed by C. Lanczos [J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163], and by other authors in the same volume. *G. E. Forsythe.*

***Weinstein, Alexander. Variational methods for the approximation and exact computation of eigenvalues.** Simultaneous linear equations and the determination of eigenvalues, pp. 83–88. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

This paper (presented in August 1951 at a Symposium held by the National Bureau of Standards Institute for Numerical Analysis) reviews the method of intermediate problems of Weinstein and Aronszajn, and gives some new results of Weinberger first published subsequently.

Let L be a positive definite completely continuous symmetric operator on a real Hilbert space $\mathbf{H}$. Let L have the sequence of known eigenvalues $\lambda_n^{(0)}\downarrow 0$ and corresponding eigenvectors $u_n^{(0)}$. Let $\mathbf{Q}$ be a closed linear subspace of $\mathbf{H}$ and let $\mathbf{P}=\mathbf{H}\ominus\mathbf{Q}$. Let L' be the projection of L into $\mathbf{Q}$, with unknown eigenvalues λ_n'. Let $p_1, p_2, \cdots$ be a basis for $\mathbf{P}$, and let $\mathbf{H}_m=\mathbf{H}\ominus\{p_1, \cdots, p_m\}$. If we call $L^{(m)}$ with ordered eigenvalues $\lambda_n^{(m)}$ the projection of L into $\mathbf{H}_m$, it is known that $\lambda_n^{(m)}\downarrow\lambda_n'$ as $m\uparrow\infty$. Moreover, the $\lambda_n^{(m)}$ are explicitly computable in terms of the $\lambda_n^{(0)}$.

Weinberger [Proc. Amer. Math. Soc. **3**, 643–646 (1952); these Rev. **14**, 290] shows that, if p_n is the projection of $u_n^{(0)}$ into $\mathbf{P}$, then $\lambda_n^{(m)}\geqq\lambda_n'\geqq\lambda_n^{(m)}-\lambda_{m+1}^{(0)}$, for $n=1, 2, 3, \cdots$. Weinberger [Pacific J. Math. **2**, 413–418 (1952); these Rev. **14**, 290] shows that p_1 can be so chosen that the weaker of the following known inequalities becomes an equality: $\lambda_n^{(1)}\geqq\lambda_{n+1}^{(0)}$ or $\lambda_n^{(1)}\geqq\lambda_n'$.

Some of the classical Sturm-Liouville separation theory is interpreted briefly and extended. *G. E. Forsythe.*

***Fichera, Gaetano. On general computation methods for eigenvalues and eigenfunctions.** Simultaneous linear equations and the determination of eigenvalues, pp. 79–82. National Bureau of Standards Applied Mathematics Series, No. 29. U. S. Government Printing Office, Washington, D. C., 1953. $1.50.

Two methods of approximating eigenvalues of nonsymmetric operators in Hilbert space are expounded briefly; they have been given by M. Picone and applied at the Italian Institute for the Applications of Calculus. First method: let $L[u]$ and $M[u]$ be two bounded linear operators defined on a separable Hilbert space $\mathbf{H}$. The given eigenvalue problem is $L[u]+\lambda M[u]=0$, $\|u\|=1$. It is solved by u, λ such that $P(u, \lambda)=\|L[u]+\lambda M[u]\|^2=0$, $\|u\|=1$. Let $\{v_k\}$ be a basis of $\mathbf{H}$, and let $u^{(n)}=\sum_{k=1}^n c_kv_k$. Let $P_n=P(u^{(n)}, \lambda)$ and let $Q_n=\|u^{(n)}\|^2$; these are Hermitian forms in the c_k. Hence to find the stationary values of P_n subject to $Q_n=1$ leads to a finite eigenvalue problem with n real roots $\mu=\mu(n, \lambda)$. For fixed n, let $\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_{\nu_n}^{(n)}$ be the minimizing points for the n functions $\mu(n, \lambda)$. These complex numbers $\lambda_i^{(n)}$ are taken as the nth approximation of the first ν_n eigenvalues λ_i of the given eigenvalue problem in $\mathbf{H}$. This first method has only an empirical justification, but has proved useful for both eigenvalues and eigenfunctions.

The second method is for systems of linear ordinary differential equations of type (*) $dx_h(t)/dt=\sum_{k=1}^p H_{hk}(t, \lambda)x_k(t)$ $(h=1, 2, \cdots, p)$ with certain homogeneous boundary conditions, where the H_{hk} are continuous. The second method is based on a Cauchy-Lipschitz approximation method for solving (*), and has been justified by T. Viola [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **29**, 180–185 (1939)]. Both methods are applied numerically to a boundary-value problem for an ordinary differential operator of order four. *G. E. Forsythe* (Los Angeles, Calif.).

Flood, Merrill M. On the Hitchcock distribution problem. Pacific J. Math. **3**, 369–386 (1953).

The Hitchcock distribution problem (or transportation problem) [J. Math. Physics **20**, 224–230 (1941); these Rev. **3**, 11] asks for a set of mn non-negative numbers x_{ij} subject to $\sum_i x_{ij}=c_j$ and $\sum_j x_{ij}=r_i$ for $i=1, \cdots, m$ and $j=1, \cdots, n$ and minimizing $\sum_{i,j}x_{ij}d_{ij}$, where r_i, c_j, and d_{ij} are given positive numbers subject to $\sum_j c_j=\sum_i r_i$. This paper proposes a computational procedure applicable when the given data are integers. The method is too complicated to be summarized here; it is combinatorial in nature and uses certain information gained by dualizing the problem. A bibliography lists references to the origins of the problem and to other computational schemes. *H. W. Kuhn.*

Best, G. C. A minimum problem solved by mesh methods. Math. Tables and Other Aids to Computation **8**, 11–13 (1954).

In the following, the function $y(x)$, subject to

$$y(0)=y(1)=1,$$

is sought which will minimize the integral

$$I=\int_0^1 y^{-1}(1+y'^2)^{\frac{1}{2}}dx.$$

This problem can be solved by the usual methods of the calculus of variations but the differential equation involved is rather complicated. It is proposed here to solve the problem by mesh or "assumed polynomial" methods.
Extract from the paper.

Morduchow, Morris. Integrals and equal division sums. Math. Mag. **27**, 65–68 (1953).

Lozinskiĭ, S. M. Estimate of the error of an approximate solution of a system of ordinary differential equations. Doklady Akad. Nauk SSSR (N.S.) **92**, 225–228 (1953). (Russian)

Let the column vector $y(t)$ satisfy $y'=f(t, y)$ (where the prime signifies differentiation); for arbitrary $Y(t)$ of class C' let $\tau=Y'-f(t, Y)$, $\Delta=Y-y$; let $U(t)$ be a nonsingular matrix (possibly complex) of class C'; $J(t, y)$ the Jacobian matrix of $\partial f_i/\partial y_j$; $Q=U^{-1}JU-U^{-1}U'$, $\zeta=U^{-1}\tau$, $\theta=U^{-1}\Delta$. All definitions apply to some interval $t_0\leqq t\leqq T$ and some

convex region in y-space. Let the real, continuous matrix $A(t)$ of positive elements be such that its diagonal elements exceed the real parts of the diagonal elements of Q and its off-diagonal elements exceed the moduli of the off-diagonal elements of Q. Then if $\epsilon' = A\epsilon + |\zeta|$, where $|\zeta|$ is the vector whose elements are the moduli of those of ζ, and $|\theta(t_0)| \leqq \epsilon(t_0)$, then also $|\theta(t)| \leqq \epsilon(t)$ for $t_0 \leqq t \leqq T$ and $|\Delta(t)| \leqq |U(t)|\epsilon$ (the inequality signs relate all pairs of corresponding elements). The theorem is stated without proof, as are three others which, in effect, interpret this first theorem in terms of different types of norm.

A. S. Householder (Oak Ridge, Tenn.).

Du Fort, E. C., and Frankel, S. P. Stability conditions in the numerical treatment of parabolic differential equations. Math. Tables and Other Aids to Computation **7**, 135–152 (1953).

The numerical stability of finite-difference representations of the diffusion equation is discussed. A generally stable difference equation is exhibited and illustrative examples worked out. *H. Polachek* (Washington, D. C.).

Blanch, Gertrude. On the numerical solution of parabolic partial differential equations. J. Research Nat. Bur. Standards **50**, 343–356 (1953).

This paper discusses the solution by finite-difference methods of a two-dimensional parabolic partial differential equation containing a non-linear term. Various finite-difference representations are considered in detail, and the author seeks to determine in each case the mesh ratio which will require the least amount of work. It is found that the largest admissible mesh ratio is not necessarily the most economical. In the last section a generalization is given to a method of Hartree and Womersly [Proc. Roy. Soc. London. Ser. A. **161**, 353–366 (1937)] for improving a solution from two difference approximations. *H. Polachek*.

Schäfke, Friedrich Wilhelm. Verbesserte Konvergenz- und Fehlerabschätzungen für die Störungsrechnung. Z. Angew. Math. Mech. **33**, 255–259 (1953).

The eigenvalue problem $Fy + \lambda y + \mu G y = 0$ with μ as perturbation parameter is considered. F and G are linear operators. If λ_m is an eigenvalue of the unperturbed problem $Fy + \lambda y = 0$, then a power expansion

$$\lambda_m(\mu) = \lambda_m + \lambda_{m_1}\mu + \lambda_{m_2}\mu^2 + \cdots$$

gives under certain conditions the eigenvalues of the perturbed problem. The author's aim is to give a greater lower bound for the radius of convergence of $\lambda_m(\mu)$ than hitherto known by previous publications. He assumes: R to be a complex linear space with a positive definite Hermitian scalar product (f, g) for its elements $f, g, \cdots$; U and U^* to be subspaces of R; F and G to be defined in U and another linear operator F^* to be defined in U^*, such that $(Fu, u^*) = (u, F^*u^*)$ with $u \in U$ and with $u^* \in U^*$. A continuous and real-valued function $\gamma(\xi, \eta)$ shall exist for $0 \leqq \xi$, $\eta < \infty$ such that $\gamma(\xi, \eta) > 0$ for ξ, $\eta > 0$, $\gamma(\zeta\xi, \zeta\eta) = \zeta\gamma(\xi, \eta)$ and $\gamma(\xi_1, \eta_1) \leqq \gamma(\xi_2, \eta_2)$ for $\xi_1 \leqq \xi_2$; $\eta_1 \leqq \eta_2$. This function shall satisfy the inequality $(Gu, Gu)^{\frac{1}{2}} = \|Gu\| \leqq \gamma(\|u\|, \|Fu\|)$. The eigenvalue pairs λ, μ shall be determined by either $\Delta(\lambda, \mu) = 0$, or, if $\lambda \neq 0$, by $\Delta(\lambda^{-1}, \mu) = 0$, where Δ denotes a suitable entire function in both of its variables. If $Fy_n + \lambda_n y_n = 0$ has a nontrivial solution, then $F^*y_n^* + \lambda_n y_n^* = 0$ shall have also one. Furthermore, it is assumed that $(y_m, y_n^*) = \delta_{mn}$ and $\beta^2 \sum_n |(f, y_n^*)|^2 \leqq (f, f) \leqq \alpha^2 \sum_n |(f, y_n^*)|^2$ with $0 < \beta \leqq \alpha < \infty$ for any $f \in R$.

The following theorem is established: Let λ_m be a single eigenvalue of the unperturbed problem. Then the power expansion $\lambda_m(\mu)$ converges at least for

$$|\mu| < \beta\alpha^{-1}[\gamma(2d_m^{-1}, D_m)]^{-1}$$

with $d_m = \inf_{\lambda_n \neq \lambda_m} |\lambda_m - \lambda_n|$, $D_m = \max_\lambda \sup_n |\lambda_n/(\lambda - \lambda_n)|$ with $|\lambda - \lambda_m| = \frac{1}{2}d_m$; $|\lambda_m(\mu) - \lambda_m| < \frac{1}{2}d_m$ for these values of μ. An example related to Mathieu's differential equation is given. *H. Bückner* (Schenectady, N. Y.).

Bernasconi, Angela. Su un metodo per l'integrazione approssimata dell'equazione di Dirac. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **15**(84), 261–271 (1951).

Die von G. L. Sewell angegebene Methode [Proc. Cambridge Philos. Soc. **45**, 631–637 (1949)] zur näherungsweisen Integration der Diracschen Gleichung wird für den Fall erweitert, dass noch zusätzlich ein von den Kernkräften herrührendes skalares Potential in den Energieausdruck eintritt. *P. Funk* (Wien).

Watkins, Charles E., Runyan, Harry L., and Woolston, Donald S. On the kernel function of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. NACA Tech. Note no. **3131**, 44 pp. (1954).

The kernel to Küssner's [Luftfahrtforshung **17**, 370–378 (1940) = NACA Tech. Memo. no. **979** (1941); these Rev. **2**, 330, 331] integral equation for the title problem is placed in a form suitable for numerical calculation by isolating the singular portion. The sonic problem is treated as a special case. *J. W. Miles* (Los Angeles, Calif.).

Yuškov, P. P. Estimate of the error in the approximate values of the Fourier coefficients when substituting a broken line or set of parabolas for a given curve. Akad. Nauk SSSR. Inženernyĭ Sbornik **14**, 204–210 (1953). (Russian)

In earlier papers [same Sbornik **6**, 197–210 (1950); **10**, 213–222 (1951); these Rev. **13**, 288, 874] the author presents formulas for the approximate evaluation of Fourier coefficients. To secure the approximation he replaces the function to be expanded by a concatenation of polynomial arcs. The present paper gives expressions for the error of the computed coefficients in the cases of polynomial arcs of degree one, two or three. *W. E. Milne* (Corvallis, Ore.).

Salzer, John M. Frequency analysis of digital computers operating in real time. Proc. I. R. E. **42**, 457–466 (1954).

Carter, W. C., and Ellis, M. A comparison of order structures for automatic digital computers. J. Operations Res. Soc. Amer. **2**, 41–58 (1954).

Knight, L. Valve reliability in digital calculating machines. Electronic Engrg. **26**, 9–13 (1954).

Clippinger, R. F., Dimsdale, B., Levin, J. H. Automatic digital computers in industrial research. II. J. Soc. Indust. Appl. Math. **1**, 91–110 (1953).

For part 1 see same J. **1**, 1–15 (1953); these Rev. **15**, 167.

Johnson, Walter E. An analogue computer for the solution of the radio refractive-index equation. J. Research Nat. Bur. Standards **51**, 335–342 (1953).

MECHANICS

Čerkudinov, S. A. Method of multiple interpolation in the synthesis of mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 40, 5–48 (1951). (Russian)

Rotinyan, L. A. Dynamic synthesis of double-cam mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 40, 62–84 (1951). (Russian)

Rešetov, L. N. Application of the circular arc for profiling a cylindrical cam with a rolling rotating pusher. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 40, 85–97 (1951). (Russian)

Nikolaev, E. N. Graphical method for the determination of the resultant vector of the inertial forces of a mechanism. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 40, 98–101 (1951). (Russian)

Kucenko, S. M. Application of the method of best approximation of Čebyšev functions to the calculation of counterweights of locomotive machines. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 41, 5–15 (1951). (Russian)

Abramov, B. M. Motion of a rigid body with connections with friction. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 41, 16–35 (1951). (Russian)

Sereda, V. T. Investigation of the errors of mechanisms of higher classes and orders. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 41, 43–60 (1951). (Russian)

Geronimus, Ya. L. Approximate method for the calculation of a balanced counterweight. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 41, 61–66 (1951). (Russian)

Litvin, F. L. The application of the analytic geometry of bevel gears to the determination of the thickness of teeth of straight-toothed gear wheels with the aid of balls. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 44, 5–21 (1952). (Russian)

Kolčin, N. I. Computation formulas for the method of checking the thicknesses of the teeth of bevel gears with the aid of balls. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 44, 22–28 (1952). (Russian)

Kudryavcev, V. N. The synthesis of a transmission having a transmission ratio near to unity and with a minimal loss in friction. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 44, 29–38 (1952). (Russian)

Bat', M. I. The equation of motion of a plane bar mechanism with an elastic intermediate link. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 44, 39–50 (1952). (Russian)

Moroškin, Yu. F. On forms of the basic equations of the geometry of mechanisms. Doklady Akad. Nauk SSSR (N.S.) 91, 745–748 (1953). (Russian)

This incontinent paper, as does a previous one [same Doklady (N.S.) 82, 533–536 (1952); these Rev. 13, 697], deals not with mechanisms but with unramified spatial chains of rigid links. It announces a "solution" (in the Pickwickian sense of the word) of the "configuration problem". Stripped of bombastic verbiage ("the ultimately laconical form of the equations . . .") and of its largely ornamental welter of indices, the paper reveals that the position of a link B with respect to the preceding one A can be described by a matrix $[B/A]$ (this is not the author's notation) such that $[C/A]=[C/B][B/A]$. This, coming from a neighborhood which produced Dimentberg's ingenious and intricate work [The determination of the positions of spatial mechanisms . . . , Izdat. Akad. Nauk SSSR, Moscow, 1950; these Rev. 12, 867] on just a special case of a spatial four-bar linkage, shows the author's nice sense of the grotesque. *A. W. Wundheiler* (Chicago, Ill.).

Myard, Francis. Rotations inverses, ou differentielles, par interférence de mouvements sinusoïdaux. C. R. Acad. Sci. Paris 238, 201–202 (1954).

L'auteur montre comment, sans engrenages, obtenir l'inversion des mouvements rotatifs, ou leur demi-vitesse; et, d'une façon générale, des différentiels mécaniques satisfaisant au cas particulier: $(w-u)/(w'-u)=-1$ de la relation de Willis. *Résumé de l'auteur.*

Bagchi, Hari Das. Note on an equi-momental complex of a rigid body. J. Math. Physics 32, 307–311 (1954).

The straight lines, fixed in a rigid body, such that the moment of inertia of the body about any one of the lines has a fixed value form a quadratic complex. In this note the author discusses properties of this complex, particularly those relating to the loci described by a point A when the cone of the complex associated with A is required to have various prescribed properties. It is shown, for instance, that the locus of points A for which the associated cones admit triads of perpendicular generators is a sphere.
L. A. MacColl (New York, N. Y.).

Slomyanskiĭ, G. A. On integration of the equations of motion of a symmetric astatic gyroscope. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 411–422 (1953). (Russian)

Consider a symmetric astatic gyroscope. Let $OXYZ$ be a fixed right-hand orthogonal trihedral whose Z-axis is directed upwards along the axis of the outer gimbal bearings. Let $Oxyz$ be a moving trihedral with the z-axis along the axis of the rotor and with the x-axis along the axis of the inner gimbal bearings. The position of the z-axis is determined by two angles α and β, where β is the angle between the z-axis and its projection on the XY-plane, and α is the angle between this projection and the Y-axis. The basic problem is to express these angles α, β as functions of the time t. If the components of a torque applied are M_x, M_y, $M_z=0$, then the differential equations which govern the angles α, β are

$$(1)\qquad \ddot{\alpha}\cos\beta-2\dot{\alpha}\dot{\beta}\sin\beta+n\dot{\beta}=n^2m_y,$$
$$\ddot{\beta}-n\dot{\alpha}\cos\beta+\dot{\alpha}^2\sin\beta\cos\beta=-n^2m_x,$$

where $n=H/A$, $m_x=M_x/An^2$, $m_y=M_y/An^2$, and where $H(=\text{const.})$ and A denote the component of the kinetic moment along the z-axis and the equatorial moment of inertia of the gyroscope respectively. The general solution of (1) by quadratures is given and discussed in the following

cases: (i) $M_x = M_y = 0$; (ii) $M_z = \text{const.}$, $M_y = 0$; (iii) subcase of regular precession for $M_z = \text{const.}$, $M_y = 0$; (iv) $M_z = M_z(\beta)$, $M_y = 0$. *E. Leimanis* (Vancouver, B. C.).

Bakke, F., Olsen, H., Wergeland, H., and Øveraas, H. Note on the integration of Hamiltonian equations. Norske Vid. Selsk. Forh., Trondheim **26**, 51–53 (1953).

The authors discover that if F is a function of the canonical variables, $\dot{F} = \Pi F$ where Π is a linear differential operator. Hence, $F = F_0 \exp(t\Pi)$, assuming convergence and forgetting that H should not depend on t. If H is a polynomial in two variables q and p, the integrals of the canonical equations satisfy $H = \text{const.}$: a method for uniformization of algebraic functions. This notation is not that of the paper which also uses some words not mentioned here.

A. W. Wundheiler (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Bilimovitch, A. Sur l'homogénéisation des équations de nature vélocidique. Acad. Serbe Sci. Publ. Inst. Math. **5**, 29–34 (1953).

With special reference to an equation he had established earlier [same Publ. **2**, 37–52 (1948); these Rev. **10**, 517] and whose dimensional inhomogeneity was remarked and put to use by the reviewer [J. Washington Acad. Sci. **40**, 313–317 (1950); these Rev. **12**, 449], the author notes that it is possible to add quantities of different but related dimensions by the artifice of introducing a special choice of units. *C. Truesdell* (Bloomington, Ind.).

Gibellato, Silvio. Determinazione delle velocità indotte da un sistema di p vortici elicoidali variabili sinusoidalmente e dal sistema vorticoso associato. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **86**, 340–361 (1952).

The system of vortices is that ideally left in the wake, after an infinite time, by a propeller of p thin blades. The determination of the velocity potential of the induced field is reduced to the calculation of the potential ϕ produced by a distribution of sources on a helix, since from ϕ the appropriate doublet distribution can be found. Then ϕ is determined as an infinite series, and the mean linear and angular momentum of the induced field are calculated.

L. M. Milne-Thomson (Greenwich).

Lakshmana Rao, S. K. The motion of four rectilinear vortex filaments. Proc. Indian Acad. Sci. Sect. A. **38**, 143–147 (1953).

Having studied in a previous paper [same Proc. **34**, 250–262 (1951)] the fixed configurations of four parallel rectilinear vortex filaments (configurations in which the quadrilateral formed by the vortices has fixed sides and diagonals), the author now discusses the motion of four vortices of strengths K_1, K_2, K_1, K_2 which lie (initially and therefore permanently) at the corners of a parallelogram, those of equal strength being at opposite corners. He investigates the changes in shape of the parallelogram, using as trilinear coordinates $x_1 = a/s$, $x_2 = b/s$, $x_3 = c/s$, where $s = a+b+c$ and a, b, c are two sides of the parallelogram and a diagonal.

J. L. Synge (Dublin).

Fox, J. L., and Morgan, G. W. On the stability of some flows of an ideal fluid with free surfaces. Quart. Appl. Math. **11**, 439–456 (1954).

This is a condensed version of a technical report [Grad. Div. Appl. Math., Brown Univ., Tech. Rep. no. **2** (1952)] dealing with the stability of various flows arising in the Helmholtz-Kirchhoff free-boundary theory. The discussion is based on a theory, due to Ablow and Hayes [ibid. no. **1** (1951)], of the small perturbations of a steady, plane flow of a perfect fluid in the presence of free surfaces (i.e. surfaces of constant pressure and therefore, since external forces are neglected, of constant velocity magnitude).

The complex velocity potential $f(z)$ of a basic flow is altered by the time dependent perturbation $f_2(w, t)$, where f_2 is an analytic function of the complex velocity $w = df/dz$. The perturbed flow must then satisfy the usual conditions on the fixed boundaries, while the pressure on the perturbed free surface must remain constant. The difficulty in the problem arises from the complicated restriction the latter condition places on f_2. By the separation of variables $f_2 = G_1(w)e^{\lambda t} + G_2(w)e^{\bar{\lambda} t}$, the stability problem becomes that of determining the eigenvalues λ consistent with the above conditions.

Four cases are treated in detail, involving in each case a painstaking analysis in which the free surface condition appears as a second order differential equation: (1) a jet impinging on a finite plate (which, in the limit, includes the infinite cavity problem for a plate immersed in a uniform stream); (2) equal and opposite jets; (3) a jet issuing from an orifice whose walls are half-lines inclined at $2\pi/n$ radians; (4) a hollow vortex bounded by cylindrical walls. Cases (1), (3), and (4) are found to be stable (except case (3) when $n=1$: the Borda mouthpiece), while (2) is unstable. These conclusions seem to the reviewer an important addition to free-boundary theory, freeing it from an old objection.

J. B. Serrin (Cambridge, Mass.).

Prem Prakash. On two dimensional superposable motions. Ganita **2**, 75–80 (1951).

This paper discusses superposable flows in the sense of Ballabh [Proc. Benares Math. Soc. (N.S.) **2**, 69–79 (1940); these Rev. **3**, 283], studying in particular the condition of superposability of certain special flows which are themselves self-superposable. *D. Gilbarg* (Bloomington, Ind.).

Ostapenko, V. N. Filtration in an almost homogeneous medium. Ukrain Mat. Žurnal **5**, 350–353 (1953). (Russian)

The author proposes to solve problems involving the equation $[k(x, y)H_x]_x + [k(x, y)H_y]_y = 0$ by assuming

$$H(x, y) = h(x, y) + \epsilon(x, y),$$

where $h(x, y)$ is harmonic and $\epsilon(x, y)$ satisfies homogeneous boundary conditions. A formidable integral equation for $\epsilon(x, y)$ is written down directly, and the matter is then dropped. *R. E. Gaskell* (Seattle, Wash.).

Sokolov, Yu. D. On a problem of the theory of unsteady motion of ground water. Ukrain. Mat. Žurnal **5**, 159–170 (1953). (Russian)

This paper contains short expositions of several approximate methods of solving the problem of water flow into a rectangular trench whose bottom rests on an infinite horizontal impermeable layer, and whose water level is suddenly lowered from $h=H$ to $h=kH$, while the surrounding soil is saturated. The author is concerned with both flow rate and the curve of depression of the water surface. In his first method, the Boussinesq equation $h_t = (hh_x)_x$ is transformed into the ordinary differential equation $uu'' + (u')^2 + 2\eta u' = 0$ through the transformation $h = ukH$, $\eta = x/2(kt)^{\frac{1}{2}}$, which is

then solved by the perturbation method. In the author's second method, the level beyond some abscissa, $x=l$, is assumed "fixed" while $h(x)$ and the flow rate are found, then $l(t)$ is determined from these, with the help of the continuity condition. His third method is similar. A Fourier series method is used to solve a second problem, similar to the first but with an impermeable vertical wall placed at $x=L$.

R. E. Gaskell (Seattle, Wash.).

Kudryašev, L. I. A generalized energy form of an integral relation of boundary-layer theory. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1953**, 1440–1443 (1953). (Russian)

Leĭbenzon [Trudy Central. Aero-Gidrodinam. no. **240**, 41–44 (1935)] et Kružilin [Žurnal Tehn. Fiz. **6**, 561–570 (1936)] ont formé l'équation de l'énergie de la couche limite, le premier pour les équations de Navier, le second pour la couche limite thermique. L'auteur généralise les résultats précédents et obtient l'équation générale de l'énergie pour une couche limite plane, en contact avec une plaque rectiligne. Les hypothèses simplificatrices adoptées sont classiques; de plus, l'auteur néglige la puissance des forces de masse et le débit de chaleur dans le sens du mouvement. *J. Kravtchenko* (Grenoble).

Schlichting, H. Die laminare Strömung um einen axial angeströmten rotierenden Drehkörper. Ing.-Arch. **21**, 227–244 (1953).

The problem indicated by the title is attacked by the momentum-integral methods of boundary-layer theory, using fourth-power polynomial expressions for the velocity profiles. The profile of the tangential velocity component has a fixed shape. A calculation procedure is worked out, making use of certain universal functions, which are plotted here. The process involves simultaneous numerical solution of the first-order differential equations describing the meridianal and tangential flows. A special treatment is required at the nose. Formulas for integrated spin moment and friction drag are worked out. As examples of the application of the method, detailed calculations are carried out for a sphere, a half-body (blunt-ended rod), and two selected streamline shapes. Among the results of these calculations it is noted that there is a slight forward shift of the separation point with increasing rate of rotation of the sphere and the streamline bodies. [For a different treatment of the same physical problem, see Illingworth, Philos. Mag. (7) **44**, 389–403 (1953); these Rev. **14**, 919.] *W. R. Sears.*

Illingworth, C. R. Boundary layer growth on a spinning body. Philos. Mag. (7) **45**, 1–8 (1954).

A body of revolution begins a screw motion, moving forward in its axial direction and spinning about that axis, at a certain instant and continues steadily in this motion thereafter. It is desired to find the effect of the spin on the initial appearance of separation. The case without spin was treated by Boltze [Dissertation, Göttingen, 1908]. The method used is one of successive approximations. To begin with, the convection and pressure-gradient terms in the laminar boundary-layer equations are neglected. Successive approximations are carried out by replacing the neglected terms by their approximate values; the second and third approximations are worked out here, at least as far as required to find the skin-friction components. As an example, a sphere is considered. Separation is found to occur first at the rear stagnation point, but it occurs earlier the greater the rate of spin. [Compare the results of Schlichting in the paper reviewed above.] *W. R. Sears* (Ithaca, N. Y.).

Dizioğlu, Bekir. Zur Theorie des Wärmeübergangés in parallelen Schmierschichten. I. Rev. Fac. Sci. Univ. Istanbul (A) **17**, 159–177 (1952). (Turkish summary)

This paper is an extension of the author's earlier work [same vol., 61–65 (1952); these Rev. **13**, 1000] on heat transfer to the case of a Couette flow with pressure gradient. This part of the paper concerns only the general formulation of the mathematical problem. *Y. H. Kuo.*

Dizioğlu, Bekir. Zur Theorie des Wärmeüberganges in parallelen Schmierschichten. II. Rev. Fac. Sci. Univ. Istanbul (A) **17**, 259–281 (1952). (Turkish summary)

The author considers a heat transfer problem, neglecting the effect of dissipation. The flow field is taken to be the plane Couette flow with arbitrary wall shear, and midway between the moving walls bounding the flow there is an interface at which the velocity is continuous but the vorticity is discontinuous. The boundary-value problem is solved subject to the specified conditions at the walls and at the interface. The temperature distribution has been calculated for two shear conditions. Both tables and graphs are presented. *Y. H. Kuo* (Ithaca, N. Y.).

Nigam, S. D. The rotation of an infinite plane lamina in a viscous compressible fluid. Proc. Indian Acad. Sci. Sect. A. **38**, 116–119 (1953).

The author studies the problem for the particular case when the fluid obeys the equation of state $p=p_0+K\rho^{5/3}$, p=pressure, ρ=density, p_0, K are constants, and when the viscosity is constant. Solutions for the velocity components and the pressure are assumed in the same form as in the analysis by v. Kármán for the analogous incompressible problem, and the method of solution of the resulting equations is identical with that of v. Kármán. It is assumed that the density varies as $h^3(z)$, $h(z)$ measuring the variation of the velocity component w normal to the lamina as a function of the distance z from the lamina. This assumption together with the boundary condition on w requires the density to vanish as the lamina is approached (close to the lamina ρ varies as z^6). Curves showing the variation of the velocity components with z are presented.

The procedure aimed at rendering all variables dimensionless is erroneous, but can readily be corrected. The reviewer questions the physical validity of a solution that leads to the previously mentioned behavior of the density in the neighborhood of the lamina. *G. W. Morgan.*

★Kampé de Fériet, J. La mécanique statistique des milieux continus. Congrès International de Philosophie des Sciences, Paris, 1949, vol. III. Philosophie Mathématique, Mécanique, pp. 129–142. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

The paper discusses the possibility of the validity of the ergodic theorem for systems with an infinite number of degrees of freedom. Apart from a general discussion of certain difficulties involved especially in the case of turbulence, the author shows that the ergodic theorem holds for the case when the mechanical system is an infinite linear string. In this case the problem is relatively trivial and can be immediately reduced (as the author points out) to stationary (in the narrow sense) stochastic processes.

M. Kac (Ithaca, N. Y.).

Hopf, Eberhard. Statistical hydromechanics and functional calculus. J. Rational Mech. Anal. **1**, 87–123 (1952).

For the flow of an incompressible viscous fluid in a domain R of the 3-dimensional space x with boundaries B, the author considers as phase space Ω the function space of all vector fields $u(x)$: (a) having all derivatives required by the Navier-Stokes equations; (b) satisfying the incompressibility equation; (c) satisfying the boundary conditions. He supposes that a probability measure $P(A)$ is defined for a field of subsets $A\subset\Omega$. Assuming for the vector field $u(x,t)$ representing the flow, the semi-group property (which logically should require the proof of a uniqueness theorem for this class of flows) $u(x,t)=T^tu(x,0)$, he supposes that this probability-measure is invariant: $P(T^tA)=P(A)$. The main tool used by the author is the "characteristic functional" defined by:

$$\Phi(y,t)=\int_\Omega e^{i(y,u)}P^t(du),$$

where (y,u) denotes the scalar product of two vector fields $y(x)$, $u(x)$ and $P^t(A)=P(T^tA)$. Next, conditions are given which the characteristic-functional must satisfy; for instance, the incompressibility of the fluid mirrors itself in the following property:

$$\Phi(y+\text{grad }\varphi)=\Phi(y),$$

where φ is any scalar function in $R+B$ vanishing on B. The Navier-Stokes equations themselves correspond to a functional differential equation:

$$\frac{\partial\Phi}{\partial t}=\int_R y_\alpha(x)\Psi_\alpha(x,t)dx,$$

where the Ψ_α are some compounds of "differentials" of Φ arising from the inertia terms, friction terms and from the pressure gradient. The last part is devoted to flows of a fluid occupying the entire space, first, periodic flows where $u(x)$ is represented by a formal Fourier series: $u(x)=\sum v(k)e^{ikx}$, and next, homogeneous and isotropic flows. Some exact stationary solutions of the Φ functional equation in the case $\mu=0$ (nonviscous fluid) are given.

The reviewer wishes to point out that some assumptions made in the paper need further proof, e.g. the fact that the characteristic functional Φ completely determines the probability measure P; this has been proved for a finite-dimensional phase space Ω by Paul Levy [Calcul des probabilités, Gauthier-Villars, Paris, 1925]; as far as the reviewer is informed no proof has yet been given for a function space, even for Hilbert space. Referring to the semi-group property, the author writes: "Uniqueness has not yet been proved. The customary methods furnish this proof in the case of plane flow but, curiously enough, they fail to do so in the general three-dimensional case even with viscosity present. I find it hard to see something serious behind these mathematical difficulties." The reviewer, who introduced, several years ago, the idea of the same phase space Ω [see p. 130 of the paper reviewed above] is not quite so optimistic and is more inclined towards the views expressed by J. Leray [J. Math. Pures Appl. (9) **12**, 1–82 (1933); **13**, 331–418 (1934); Acta Math. **63**, 193–248 (1934)] that the difference between the 2 and the 3 dimensional cases plays an essential role in the theory. *J. Kampé de Fériet* (Lille).

Bass, Jean. Sur les équations fonctionnelles des fluides turbulents. C. R. Acad. Sci. Paris **237**, 645–647 (1953).

The author remarks that, when the fluid occupies the whole x space, the phase space Ω considered by E. Hopf [see the paper reviewed above] can not be a Hilbert space (the kinetic energy can not be finite) but suggests that the vector field $u(x)$ belongs to L^p for some $p<2$. The results of E. Hopf are limited to the particular case of periodic velocity fields; using Fourier-Stieltjes transforms of the vector field $u(x,t)$ the author points out that it is possible to get a useful extension of the previous results. *J. Kampé de Fériet.*

Hopf, E., and Titt, E. W. On certain special solutions of the Φ-equation of statistical hydrodynamics. J. Rational Mech. Anal. **2**, 587–591 (1953).

Applying the methods developed by E. Hopf in the paper reviewed second above, the authors consider the functional equation of the characteristic functional Φ when the x space occupied by the fluid is replaced by its Fourier dual $u(x)=\sum v(k)e^{ikx}$. For the case $\mu=0$ (nonviscous fluid) they find a solution which disagrees with the law derived by Kolmogoroff from dimensional considerations; they nevertheless prove that their solution is unique.

J. Kampé de Fériet (Lille).

Barenblatt, G. I. On the motion of suspended particles in a turbulent flow. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 261–274 (1953). (Russian)

Le mouvement des particules pesantes dans un courant turbulent peut être étudié de deux façons différentes: a) par la théorie de diffusion, les particules sont transportées par le courant mais n'ont aucune influence sur son mouvement, et b) par la théorie gravitationnelle développée par M. A. Velikanov [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1944**, no. 3] pour calculer l'influence des particules sur la dynamique du courant, ce qui permet d'évaluer le travail dépensé par le courant pour soulever les particules pesantes. Malheureusement, Velikanov a omis d'introduire ce travail dans le bilan d'énergie oscillatoire du courant.

L'auteur reprend la dernière théorie en supposant que les particules sont très petites, que l'accélération du courant est très petite par rapport à l'accélération gravitationnelle des particules, que les composantes horizontales des vitesses des particules et du courant coïncident, et en négligeant la diffusion moléculaire des particules dans le courant; on obtient un système fermé d'équations différentielles par rapport aux composantes de la vitesse, de la pression et de la densité du fluide non homogène.

Dans le présent article l'auteur introduit en outre deux conditions restrictives supplémentaires: $\rho\ll1$, $\rho(d_2-d_1)/d_1\ll1$, où ρ est le volume relatif des particules, d_1 et d_2 sont les densités constantes du liquide et des particules. L'auteur forme les équations des moyennes de la quantité du mouvement, du bilan de la masse, ainsi que l'équation de continuité, ce qui lui permet, en utilisant les idées de A. N. Kolmogorov [Izvestiya Akad. Nauk SSSR. Ser. Fis. **6**, 56–58 (1942)] de déduire l'équation du bilan de l'énergie pulsatoire du courant.

L'auteur applique ensuite les résultats obtenus au cas particulier d'un courant plan stationnaire en moyenne et homogène suivant la direction horizontale, qui présente un grand intérêt pratique. *M. Kiveliovitch* (Paris).

Sedov, L. I., Mihaĭlova, M. P., and Černyĭ, G. G. On the influence of viscosity and heat conduction on the flow of a gas behind a strongly curved shock wave. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8, 95–100 (1953). (Russian)

L'écoulement permanent avec onde de choc aval, des gaz autour d'obstacles de petites dimensions, mais de grande courbure, peut donner lieu à un jeu notable de forces de viscosité et aux phénomènes thermiques, généralement négligés dans les théories antérieures et dont les auteurs se proposent de chiffrer l'influence. A cet effet, les auteurs partent des équations: de continuité, de l'énergie, des quantités de mouvement, écrites au sommet de l'onde—dans le cas d'un mouvement plan, symétrique autour d'un axe de révolution; il est tenu compte de la viscosité et de la conductibilité thermique. Diverses hypothèses simplificatrices permettent d'en déduire les valeurs des dérivées spaciales des caractéristiques thermo-dynamiques du régime, dans le voisinage de la surface de discontinuité, tant à l'aval qu'à l'amont; les corrections dues à la viscosité et à la conductibilité en résultent. En particulier, les auteurs trouvent le rapport des températures d'arrêt; les résultats sont traduits sous forme de graphiques commodes. On conclut ainsi que, dans les conditions habituelles de l'utilisation des appareils de mesure (couples thermo-électriques), les perturbations locales peuvent être évaluées en négligeant les effets étudiés ci dessus. Mais il n'en serait pas de même pour le dispositif décrit par L. Kovásznay [J. Aeronaut. Sci. 17, 565–572, 584 (1950)]. *J. Kravtchenko* (Grenoble).

v. Krzywoblocki, M. Z. On the stability of Bénard-Kármán vortex street in compressible fluids. I. Acta Physica Austriaca 7, 283–298 (1953).

By making certain implicit linearizations, the author extends the well known stability theory of the Kármán vortex street to inviscid compressible fluids. He finds instability for all the usual vortex configurations.

D. Gilbarg (Bloomington, Ind.).

Imai, Isao. On the asymptotic behaviour of compressible fluid flow at a great distance from a cylinder in the absence of circulation. J. Phys. Soc. Japan 8, 537–544 (1953).

The author considers steady plane irrotational flow of an ideal compressible fluid past an arbitrary cylinder. Assuming uniform flow at infinity and absence of circulation, he derives asymptotic expressions for the velocity potential and stream function correct through the order $1/r^4$. This is a valuable result, and it is obtained with apparent simplicity; however, as the paper makes essential use of work not readily available [Imai, Rep. Aeronaut. Res. Inst. Tokyo Imp. Univ. no. 294 (1944)], it is difficult to check the statement made that the expansion is completely rigorous.

J. B. Serrin (Cambridge, Mass.).

Legendre, Robert. Ecoulement isentropique plan d'un fluide compressible. C. R. Acad. Sci. Paris 237, 595–597 (1953).

In an earlier paper [same C. R. 231, 1419–1421 (1950); these Rev. 12, 369] the author found a family of solutions of the hodograph equation for transonic two-dimensional isentropic flow of a gas with a certain fictitious equation of state. The results are here generalized to give a family of solutions for an arbitrary law of compressibility.

D. C. Pack (Glasgow).

Guderley, Gottfried, and Yoshihara, Hideo. Two-dimensional unsymmetric flow patterns at Mach number 1. J. Aeronaut. Sci. 20, 757–768 (1953).

The authors examine particular solutions of Tricomi's equation which are singular at one point of the sonic line, in order to find a suitable representation of the free-stream singularity for a body with a rounded nose at a small angle of attack. A procedure is outlined for obtaining the flow over a symmetrical airfoil when the flow at zero incidence is supposed to be known.

A detailed examination is made of the flow over a double-wedge airfoil at an angle of attack α sufficiently small for its effect to be given by retaining only linear terms in α. There is some difficulty in formulating the boundary conditions near the nose of the profile, but it is shown in an appendix that the assumptions introduced cannot affect the calculated lift force when only first order terms in α are significant. Details are given of the methods by which the pressure distribution over the wedge was computed and the results are shown in graphical form. *D. C. Pack* (Glasgow).

***Ludloff, H. F. On aerodynamics of blasts.** Advances in Applied Mechanics, vol. 3, pp. 109–144. Academic Press, Inc., New York, N. Y., 1953. $9.00.

The paper is a general theoretical survey of the transient pressure fields set up when uniform plane shock waves encounter obstacles of various shapes. It presents an account of the diffraction of very weak shocks around wedges of arbitrary angle, based on the theory of conical fields [A. Busemann, Schr. Deutsch. Akad. Luftfahrtforschung 7B, 105–121 (1943); these Rev. 8, 415]. The author treats next the case of a strong shock advancing over a two-dimensional obstacle whose angle of inclination to the direction of propagation remains small, following Ting and Ludloff [J. Aeronaut. Sci. 18, 143–144 (1951); 19, 317–328 (1952); these Rev. 14, 108]. He then extends this work to slender obstacles of revolution. Next he treats the head-on encounter of a shock with a wall almost perpendicular to its direction of propagation. He describes the reviewer's method and results for this problem [Proc. Roy. Soc. London. Ser. A. 200, 554–565 (1950); these Rev. 12, 370] at some length, and goes on to discuss possible alternative approaches which might yield approximate information with less analytical effort. *M. J. Lighthill* (Manchester).

Cabannes, H. Etude du départ d'un obstacle dans un fluide au repos (écoulements plans—écoulement de révolution). Recherche Aéronautique no. 36, 7–12 (1953).

When a body is started with non-zero velocity in a fluid at rest a shock-wave is instantly formed on the surface of the body. The equation of the shock-wave at later times may be expressed in the form $x=F(t)$ for one-dimensional flow and $x=F(y,t)$ for plane and axially symmetrical flow, where x is measured from the original position of the nose of the body and in the direction of motion of the body, y is measured in the perpendicular direction and t is the time from the beginning. The author finds formulae for the first two terms in the expansions of x in ascending powers of t for the three kinds of flow. For a plane or axially symmetrical body set in motion with a uniform velocity he estimates the ultimate (steady-state) distance of the shock-wave from the body by calculating the distance apart when their relative velocity vanishes. The values obtained in this way from the first two terms in the developments in series can

only be good approximations for very high speeds of the body. Comparison with experiments on spheres at Mach numbers (M) up to 1.8 shows that the theory is very badly in error in this range (a factor of 3 at $M=1.8$), but the error is seen to be decreasing steadily with mounting Mach number. *D. C. Pack* (Glasgow).

Woods, L. C. The application of the polygon method to the calculation of the compressible subsonic flow round two-dimensional profiles. Ministry of Supply [London], Aeronaut. Res. Council, Current Papers no. **115**, 27 pp. (4 plates) (1953).

A procedure for the calculation of compressible subsonic flow about two-dimensional airfoils is described in detail. Tables of required auxiliary functions as well as an illustrative example are provided in the text. The calculation of lift and moment coefficients, and their rates of change with angle of incidence is also discussed. *H. Polachek.*

Kuessner, H. G. A general method for solving problems of the unsteady lifting surface theory in the subsonic range. J. Aeronaut. Sci. **21**, 17–26, 36 (1954).

The title problem requires the solution of the Helmholtz equation

$$(1) \qquad (\nabla^2+\kappa^2)\phi=0$$

subject to the boundary conditions associated with a thin, oscillating wing in subsonic flow, viz. prescribed, normal velocity over wing plus Kutta condition along trailing edge. A formal solution is developed along classical lines, and the relevant Green's function determined by separation of variables. The author's suggestion "to put the problem on the strong shoulders of the mathematician and to develop new wave functions" overlooks the fact that (1) can be separated only in eleven coordinate systems [H. P. Robertson, Math. Ann. **98**, 749–752 (1928); L. P. Eisenhart, Physical Rev. (2) **45**, 427–428 (1934) and Ann. of Math. (2) **35**, 284–305 (1934)]. Of these, only elliptic cylinder, oblate spheroidal, and ellipsoidal coordinates lead to wing planforms (infinite ribbon, circular, and elliptic, respectively) of practical interest. These examples, none of which is new, are discussed by the author. *J. W. Miles* (Los Angeles, Calif.).

Rodriguez, A. M., Lagerstrom, P. A., and Graham, E. W. Theorems concerning the drag reduction of wings of fixed plan form. J. Aeronaut. Sci. **21**, 1–7 (1954).

This is an application of the general ideas of Hilbert space to questions of minimum drag in linearised supersonic flow. It is well-known that for a given wing plan form, the incidence and lift distributions are obtained from each other by means of certain linear operations. Unlike in most engineering applications of Hilbert space, the operators in question are not symmetrical. For any two given incidence distributions, the inner product is defined as one half the interference drag. It is then assumed that a complete set of orthogonal distributions has been constructed and the corresponding Fourier expansion for an optimal load distribution (minimum drag for given lift) is derived. It is shown that a reversed flow relation for optimal loading which had been established earlier by R. T. Jones, can be deduced also from the present theory [see R. T. Jones, same J. **18**, 75–81 (1951); these Rev. **12**, 554; cf. also E. W. Graham, ibid. **19**, 823–825 (1952)]. Another general result which is obtained in the paper under review, states that the optimal incidence distribution has zero twist. Leading edge motion is neglected in the analysis, although an interesting conjecture is stated with regard to its effect. *A. Robinson* (Toronto, Ont.).

Nonweiler, T. The theoretical wave drag at zero lift of fully-tapered swept wings of arbitrary section. Coll. Aeronaut. Cranfield. Rep. no. **76**, 42 pp. (7 plates) (1953).

The title problem requires the evaluation of a double integral involving the wing surface slope. This integral is evaluated analytically in a few particular cases, but it is concluded that, in general, it may be best computed by numerical integration. However, if $(M^2-1)\cot\Lambda<0.3$ and $A\tan\Lambda>2.5$, where M, Λ, and A denote flight Mach number, mid-chord sweepback angle, and aspect ratio, respectively, an approximate, analytical evaluation is found to be useful. *J. W. Miles* (Los Angeles, Calif.).

Šapošnikov, I. G. On the theory of convective phenomena in a binary mixture. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 604–606 (1953). (Russian)

Meixner, J. Allgemeine Theorie der Schallabsorption in Gasen und Flüssigkeiten unter Berücksichtigung der Transporterscheinungen. Acustica **2**, 101–109 (1952).

The author remarks that various previous attempts to treat simultaneously both the classical dissipative mechanisms (viscosity, heat, conduction, diffusion) and inner transformations (usually called "relaxation processes") have been fragmentary, disconnected, or faulty. He sets up the basic equations governing a two-component mixture whose concentrations are ξ and $1-\xi$; these include a Gibbsian equation of state $U=f(S, V, \xi)$, where U, S, and V are the densities of energy, entropy, and volume. The affinity is then $A\equiv-\partial U/\partial\xi$ or equivalently, the difference of the Gibbs potentials of the two components. As constitutive equations the author sets down in addition to the usual expression for the viscous stress arising from shear viscosity η only the following analogous expressions for the energy flux $\mathbf{W}$, diffusion flux $\mathbf{D}$, and the rate of production Γ of the component whose concentration is ξ:

$$\mathbf{W}=-\frac{a}{T}\operatorname{grad} T+bT\operatorname{grad}\frac{A}{T},$$

$$\mathbf{D}=-\frac{b}{T}\operatorname{grad} T+cT\operatorname{grad}\frac{A}{T},$$

$$\Gamma=\epsilon A,$$

where T is the temperature and where η, a, b, c, ϵ are empirical functions of temperature and pressure.

Next he obtains a complex frequency equation appropriate to plane infinitesimal waves. This is bi-cubic; he does not attempt a rigorous solution but calculates the first few terms in a power-series expansion. Hence he obtains formulae for absorption and dispersion which agree with or generalize others obtained by less systematic means or from less general theories. He concludes that the dissipative effects of viscosity and of inner transformations are additive at very low and at very high frequencies, but not in the intermediate range; in the former case, the effect of the latter is equivalent to that of a bulk viscosity. [Essentially the same basic theory was proposed by Z. Sakadi, Proc. Phys.-Math. Soc. Japan (3) **23**, 208–213 (1941); these Rev. **3**, 24.] *C. Truesdell* (Bloomington, Ind.).

Westervelt, Peter J. Acoustic streaming near a small obstacle. J. Acoust. Soc. Amer. 25, 1123 (1953).

Steady rotational flow is known to occur in the vicinity of a smooth rigid object oscillating to and fro in a viscous incompressible fluid. For a small oscillation amplitude the steady flow is shown theoretically to be invariant to the coordinate transformation which renders the object stationary. (From the author's summary.) *J. B. Serrin.*

Heller, G. S. Propagation of acoustic discontinuities in an inhomogeneous moving liquid medium. J. Acoust. Soc. Amer. 25, 950–951 (1953).

It is shown that a surface in a moving inhomogeneous liquid on which the time derivatives of pressure and velocity are discontinuous must propagate according to a generalized eikonal equation. The usual assumption of infinitesimal amplitudes is not required in this treatment. (Author's summary.) *D. C. Pack* (Glasgow).

Kornhauser, E. T. Ray theory for moving fluids. J. Acoust. Soc. Amer. 25, 945–949 (1953).

The generalized eikonal equation for moving fluids is obtained from a very simple physical argument. Elementary solutions for uniformly moving media are discussed. A generalized form of Snell's law is derived, and it is specialized in three separate ways to yield formulas useful for solution of problems in stratified media. Examples of its application are given in each case. (Author's summary.)
D. C. Pack (Glasgow).

Mintzer, David. Wave propagation in a randomly inhomogeneous medium. I. J. Acoust. Soc. Amer. 25, 922–927 (1953).

En se basant sur les résultats de T. H. Ellison [J. Atm. Terrest. Phys. 2, 14–21 (1951)] l'auteur établit l'équation d'onde pour la pression en supposant que l'indice de réfraction s'écarte très peu de sa moyenne. Dans ce cas on peut utiliser comme solution approchée, la première approximation déduite par Ellison. Afin de faciliter les calculs, l'auteur introduit le temps T = la langueur de l'impulsion, et après quelques réductions simples il trouve l'expression de l'amplitude moyenne de la pression $\bar{p}$ en fonction de T, de la variation de l'indice de réfraction $n(x, y, z)$ et de r = distance du point d'observation à l'origine. En prenant dans l'expression de $\bar{p}$ la moyenne de $n(x, y, z)$ suivant les différentes distributions, on obtient la valeur $\langle\bar{p}\rangle$ d'où l'on tire facilement le coefficient de variation

$$V^2 = \frac{\langle|\bar{p}|^2\rangle - |\langle\bar{p}\rangle|^2}{|\langle\bar{p}\rangle|^2}.$$

En substituant dans V^2 les expressions de $\langle|\bar{p}|^2\rangle$ et $|\langle p\rangle|^2$ on trouve finalement la loi $V^2 \sim r^3$ confirmée par les observations de M. J. Sheehy [J. Acoust. Soc. Amer. 22, 24–28 (1950)]. *M. Kiveliovitch* (Paris).

Kay, Irvin. Diffraction of pulses by parabolic cylinders and paraboloids of revolution. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-53**, i+21 pp. (1953).

The diffraction of a plane pulse by a parabolic cylinder and by a paraboloid of revolution is investigated for the boundary conditions $u=0$ and $\partial u/\partial n=0$. The pulse is assumed to be incident on the convex side in the direction parallel to the main axis of the obstacle. Using Laplace-transform technique, the author derives in each case the required solution from a Volterra integral equation, which is then solved by a process of iteration. If the parabolic cylinder degenerates into a half-plane, the solution reduces to one obtained previously [J. B. Keller and A. Blank, same Research Rep. No. **EM-21** (1950); these Rev. 12, 564; Kay, ibid. No. **EM-43** (1952); these Rev. 14, 1142]. Reference is made to Friedlander [Proc. Cambridge Philos. Soc. 38, 399–400 (1942); these Rev. 4, 170] and Chester [Quart. J. Mech. Appl. Math. 5, 196–205 (1952); these Rev. 13, 1004] who obtained results essentially identical with the author's in a different way. *C. J. Bouwkamp.*

Hwang, S. S. On the role of anticyclones in the atmosphere. Acad. Sinica Science Record 5, 139–144 (1952). (Chinese summary)

An expression for the local rate of change of the kinetic energy in a finite volume of air is obtained, which involves a term corresponding to the change in kinetic energy due to the horizontal divergence of the velocity field. Interpreting this term with the help of the tendency equation leads to the conclusion that an anticyclone is a kinetic-energy-producing system whilst a cyclone is a kinetic-energy-consuming system. Thus cyclones must acquire energy from anticyclones. *G. C. McVittie.*

Elasticity, Plasticity

***Butty, Enrique. Tratado de elasticidad teorico-tecnica. Tomo I. Teoria general. Problemas elasticos planos y espaciales.** [Treatise on theoretical and engineering elasticity. Vol. I. General theory. Plane and spatial elastic problems.] Centro Estudiantes de Ingenieria de Buenos Aires, Buenos Aires, 1946. xxxi+1004 pp.

Part I: General theory of elasticity of homogeneous and isotropic solids. Ch. I. Stress and strain of continuous media. Ch. II. Differential equations, principles, and fundamental theorems of the equilibrium and motion of homogeneous and isotropic solids. Part II: Elastic solids with plane stress or strain. Ch. III. Solution of rectangular plates in cartesian coordinates. Ch. IV. Solution in polar coordinates of infinite and circular plates. Ch. V. Solution of plates by means of functions of a complex variable and by curvilinear coordinates. Ch. VI. Graphical representation and experimental determination of plane states of stress. Photo-elasticity. Part III: Elastic solids with spatial stress and strain. Ch. VII. Criteria for the solution of spatial elastic systems with rotational symmetry. Ch. VIII. General criteria for the solution of spatial elastic systems. Ch. IX. Solution of spatial elastic systems in curvilinear coordinates. Ch. X. Spatial elastic problems. Ch. XI. Infinite and semi-infinite solids, centers of pressure, contact pressure, hardness, elastic shock, and influence of temperature.
Table of contents.

Platrier, Charles. Conditions d'intégrabilité du tenseur de déformation totale dans une transformation finie d'un milieu à trois dimensions. Ann. Ponts Chaussées 123, 703–709 (1953).

In the last part of footnote 3 to §18 of a paper by the reviewer [J. Rational Mech. Anal. 1, 125–171, 173–300 (1952); 2, 593–616 (1953); these Rev. 13, 794; 15, 178] are

cited sixteen papers from 1902 onward where the author's result is derived. Doubtless the list is not complete.
C. Truesdell (Bloomington, Ind.).

Schaefer, Hermann. Die Spannungsfunktionen des dreidimensionalen Kontinuums und des elastischen Körpers. Z. Angew. Math. Mech. **33**, 356–362 (1953). (English, French and Russian summaries)

The reviewer disagrees with the author's conclusion: "Dass unsere obige Betrachtungen . . . uns aber gerade auf die Boussinesq-Neuberschen Darstellungen . . . führen würden, konnte nicht von vornherein vermutet werden." The author indicates that he was not aware of the work of recent Russian writers on stress function; apparently neither he nor they are aware of the extensive prior Italian work, beginning with B. Finzi [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584, 620–623 (1934)]. It is to be hoped that soon there will be enough papers on this increasingly popular subject. *C. Truesdell.*

Benthem, J. P. Note on the general stress-strain relations of some ideal bodies showing the phenomena of creep and of relaxation. Nationaal Luchtvaartlaboratorium, Amsterdam. Report S. **426**, 9 pp. (1953).

This paper contains simple remarks on the constitutive equations for various domains of continuum mechanics.
C. Truesdell (Bloomington, Ind.).

Trenin, S. I. On solutions of the equilibrium equations of the axially symmetric problem of the theory of elasticity. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **8**, No. 2 7–13 (1953). (Russian)

The author lists a number of the known general solutions of the equations of linear elasticity for static, axially symmetric problems. Relations connecting the various solutions are given. *J. L. Ericksen* (Washington, D. C.).

Eubanks, R. A. and Sternberg, E. On the axisymmetric problem of elasticity theory for a medium with transverse isotropy. J. Rational Mech. Anal. **3**, 89–101 (1954).

The authors begin by proving the following generalization of a theorem due to Almansi [Ann. Mat. Pura Appl. (3) **2**, 1–51 (1899)]: Let R be a region of the (ρ, z)-plane such that a straight line parallel to the z-axis intersects the boundary of R in at most two points. Let $F_n(\rho, z)$ be a solution of $\prod_{i=1}^{n} V_i^2 F_n = 0$ in R, where

$$V_i^2 = \sum_{j=0}^{r} p_j(\rho)\partial^j/\partial\rho^j + c_i^2\partial^2/\partial z^2,$$

the c_i are constants, and the functions $p_i(\rho)$ are continuous in R with $p_r \neq 0$. Then $F_n = F_{n-1} + z^m F^{(n)}$, where

$$\prod_{i=1}^{n-1} V_i^2 F_{n-1} = V_n^2 F^{(n)} = 0,$$

and m is the number of the coefficients c_i^2 $(i = 1, \cdots, n-1)$ which are equal to c_n^2.

Directing their attention to the equations of equilibrium of linear elasticity for materials possessing transverse isotropy, they use the theorem above to show that the solution to any problem with torsionless axisymmetry can be expressed in terms of two harmonic functions. Conversely, any pair of harmonic functions determines such a solution. This analysis provides a completeness proof for Lekhnitsky's solution of these equations [Akad. Nauk SSSR. Žurnal Prikl. Mat. Meh. (N.S.) **4**, no. 3, 43–60 (1940)].
J. L. Ericksen (Washington, D. C.).

Kalandiya, A. I. Solution of some problems on the bending of an elastic plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 293–310 (1953). (Russian)

The author considers the problem of the determination of the deflection of a thin isotropic elastic plate with free edges. In the first part of the paper an effective method of solution is given when the domain occupied by the plate can be mapped conformally onto the unit circle by means of a polynomial. This particular problem had been dealt with earlier by A. I. Lur'e [Izvestiya Leningrad Politehn. Inst. Kalinin. **31**, 305–319 (1928)]. By way of examples, the plane with an elliptical and a circular hole are treated. The second part is concerned with the same problem for an elliptical plate, which is reduced to the solution of an infinite system of linear equations for infinitely many unknown coefficients. The existence of a bounded solution of this system follows from the general theory of regular systems [Kantorovich and Krylov, Approximate methods of higher analysis, 3rd ed., Moscow-Leningrad, 1950; these Rev. **13**, 77]. The uniformly loaded elliptic plate with free edges is treated numerically, and the value of the deflection at the center is near to that given by B. G. Galerkin [Elastic thin plates, Gostroĭizdat, Leningrad-Moscow, 1933].
J. B. Dias (College Park, Md.).

Sengupta, H. M. On the bending of an elastic plate. IV. Bull. Calcutta Math. Soc. **45**, 9–19 (1953).

[For parts I, II, and III, see same Bull. **41**, 163–172 (1949); **43**, 123–131 (1951); **44**, 111–123 (1953); these Rev. **11**, 287; **13**, 1005; **15**, 75.] The author obtains an infinite series representing the deflection of the central plane of a thin elliptic plate clamped at the edge and subject to a uniformly distributed load applied normal to a rectangular region on one face of the plate. *J. L. Ericksen.*

Illing, Edith. The bending of thin anisotropic plates. Quart. J. Mech. Appl. Math. **5**, 12–28 (1952).

The complex variable method is used to discuss the bending of a thin orthotropic plate, with special reference to a clamped circular plate subjected to constant, linear and quadratic transverse load-distributions. No numerical results are given, but it is found that anisotropy effects appear only for quadratic and higher order load-distributions.
B. R. Seth (Kharagpur).

Reissner, Eric. On finite twisting and bending of circular ring sector plates and shallow helicoidal shells. Quart. Appl. Math. **11**, 473–483 (1954).

Verf. betrachtet eine dünne Ringsektorplatte unter der Einwirkung von zwei gleich grossen und entgegensetzt gerichteten Kräften senkrecht zur Plattenebene längs der Achse durch den Mittelpunkt des Ringes. Die Ringsektorplatte mag als ein Teil der Windung einer eng gewundenen Schraubenfeder betrachtet werden. Die Spannungsverteilung in einem verdrehten Ringsektor von Rechteckquerschnitt wurde zuerst von J. H. Michell [Proc. London Math. Soc. **31**, 130–146 (1889), p. 140] als eine Aufgabe der räumlichen Elastizitätstheorie aufgefasst. Eine Lösung dieses Problems mit Hilfe der Theorie dünner Platten wird sich zur Lösung von Michell verhalten, wie die Lösung für die Torsion einer Rechteckplatte von Thomson und Tait zur Lösung von St. Venant für die Torsion eines Stabes von Rechteckquerschnitt. Ein Resultat dieser Arbeit ist die weitere Beobachtung, dass die Handhabung des Problems als Plattenproblem es möglich macht, nichtlineare Effekte

in relativ einfacher Weise unter Anwendung der Gleichungen für endliche Durchbiegung dünner Platten zu berechnen. Ausser dem Problem der nicht linearen Effekte der ursprünglich ebenen Platte wird das entsprechende Problem für eine seichte schraubförmige Schale untersucht, wobei Aufschlüsse über die Wirkung einer anfänglichen Steigung der Schalenfläche auf Spannungen und Formänderungen erhalten werden. Bei Betrachtung dieser beiden Probleme wird Verf. auf die Untersuchung der gleichzeitigen Wirkung eines Paares axialer Kräfte wie oben angegeben und eines Momentenpaares geführt, dessen Drehachse mit der Richtung dieser Kräfte zusammenfällt. Insbesondere wird ermittelt, welche Grösse die Momente annehmen müssen, um der Kopplung tangentialer Verschiebungen der Ringplatte mit axialen Verschiebungen infolge der axialen Kräfte vorzubeugen. Es ergibt sich, dass wenn diesen tangentialen Verschiebungen vorgebeugt wird, die Wirkung der Nicht-Linearität sowie der Einfluss einer anfänglichen Steigung viel ausgeprägter wird als bei Abwesenheit der Momente. Weiter entsteht ein Problem der elastischen Instabilität der Sektorringplatte, die mit der Anwesenheit des Momentpaares verbunden ist. Als ein weiteres Resultat ergibt sich, dass Nicht-Linearität sich relativ stärker in Bezug auf die Grösse der Spannungen als auf die Beziehungen zwischen axialen Kräften und Verschiebungen bemerkbar macht. Ein ähnliches Ergebnis bezüglich der endlichen Torsion dünner Rechteckplatten liegt bereits vor [S. Timoshenko, Strength of materials, vol. II, Van Nostrand, New York, 1941, S. 301]. Abgesehen vom Verfahren unterscheidet sich die vorliegende Behandlung der Sektorringplatte von der Theorie dünner Stäbe in folgender Beziehung: Es ist auf die nicht lineare Wirkung Rücksicht genommen, die von Bedeutung ist, wenn das Verhältnis Breite: Dicke des Querschnitts hinreichend gross ist. Bei der schraubenförmigen Schale wird diese Wirkung sogar im Gebiete der Anwendbarkeit der linearen Theorie als vorhanden nachgewiesen. In der Theorie der Verdrehung rechteckiger Platten bestätigt sich diese Wirkung von selbst, indem die axialen Normalspannungen dem Quadrat des Verdrehungswinkels proportional sind. Diese Normalspannungen haben keine axialen Resultanten oder Biegemomente, aber verursachen ein Drehmoment, das sich der dritten Potenz des Verdrehungswinkels proportional ergibt [A. E. Green, Proc. Roy. Soc. London. Ser. A. **154**, 430–455 (1936); **161**, 197–220 (1937)]. Ein entsprechendes Ergebnis wird hier beim Problem des Ringsektors erhalten. Während die Resultate der Theorie dünner Stäbe auf der Annahme fusst, dass das Verhältnis zwischen der Breite des Querschnitts und dem Radius der Mittellinie des Querschnitts hinreichend klein ausfällt, gelten die vorliegenden Ergebnisse unabhängig von diesem Verhältnis.

R. Gran Olsson (Trondheim).

Higuchi, Masakazu. Orthotropic semi-infinite plate with a hole. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **2**, 161–163 (1953).

Using methods and results from a previous paper [same Rep. **1**, 33–45 (1952); these Rev. **13**, 1005], the author gives expressions for the stress function for an orthotropic semi-infinite plate with an elliptic hole. The solution is not in closed form, but involves an infinite number of unknown constants which are derivable from linear simultaneous equations. Numerical calculation is possible in terms of Bessel's function of order m in the case when both the periphery of the hole and the straight edge of the plate are free and the plate is under uniform tension parallel to the edge. It is stated also that similar equations may be derived in the other cases when the plate is free on the boundaries and under a uniform shearing force or a bending force. A diagram indicating the stress concentration in the case of a circular hole near the edge of a plate under uniform tension is given.

R. M. Morris (Cardiff).

Müller, W. Die Momentenfläche der elastischen Platte oder Pilzdecke und die Bestimmung der Durchbiegung aus den Momenten. Ing.-Arch. **21**, 63–72 (1953).

This paper is complementary to the author's earlier investigation [Ing.-Arch. **20**, 278–290 (1952); these Rev. **14**, 925] of the deflections of rectangular plates and of plates on periodically arranged point supports according to the classical linear theory. In the present paper the bending moments for a simply supported rectangular plate under an arbitrarily located concentrated load, and for a uniformly loaded plate on point supports, are determined explicitly in terms of Theta functions. Contour lines for the corresponding moment surfaces are given.

E. Sternberg.

Èstrin, M. I. On a method of solution of a homogeneous problem for a symmetrically loaded torus-shaped shell. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 619–622 (1953). (Russian)

Prelog, Ervin. Elastostatik der dicken Zylinderschalen. Acad. Serbe Sci. Publ. Inst. Math. **5**, 115–132 (1953).

Verf. geht von den Laméschen Bedingungen der Kompatibilität aus, die im dreidimensionalen Fall vektoriell geschrieben werden können

$$(1)\qquad (\lambda+2\mu)\,\mathrm{grad}\,\epsilon-2\mu\,\mathrm{curl}\,\omega+\rho\bar{P}=0,$$

wo λ und μ die Laméschen Konstanten, ϵ die räumliche Dilatation, ρ die spezifische Masse, $\bar{P}$ die äussere Kraft, $\bar{v}$ den Verschiebungsvektor, ω die Rotation d.h. (2) $\bar{\omega}=\frac{1}{2}\,\mathrm{curl}\,\bar{v}$ [vgl., z. B., A. Sommerfeld, Mechanics of deformable bodies, vol. II, Academic Press, New York, 1950, S. 61; diese Rev. **11**, 700]. Diese vektorielle Form der Laméschen Gleichungen gestattet einen sehr einfachen Übergang auf krummlinige Koordinaten, der zunächst angegeben wird, wonach die Gleichungen auf Zylinderkoordinaten transformiert werden. Die Integration geschieht durch Reihenansätze die in Bezug auf den Winkel φ und die Erzeugende z ("Generator") nach trigonometrischen Funktionen fortschreiten, während sich danach für die Verschiebungen in den drei ausgezeichneten Richtungen simultane Differentialgleichungen mit dem Radius r als unabhängig Veränderliche ergeben, die nach dem Verfahren von Frobenius durch Reihen gelöst werden. Verf. behandelt danach einige Sonderfälle, wobei der erste Fall einer in ihrer Ebene belasteten Ringscheibe, der zweite Fall dem Spannungszustand in einer dicken Kreisplatte entspricht [siehe S. Woinowsky-Krieger, Ing.-Arch. **4**, 203–226, 305–331 (1933)]. Beim dritten Belastungsfall wird die Zylinderschale so belastet und gestützt angenommen, dass die Belastungen sowie die Verschiebungen vom Winkel φ unabhängig sind. Solche Fälle liegen bei einer Schale vor die auf reine Torsion beansprucht wird, ferner bei einem Zylinder der ringsum stückweise auf Zug oder Druck beansprucht wird, wofür Verf. ein numerisches Beispiel durchrechnet.

R. Gran Olsson (Trondheim).

Hoff, N. J., Kempner, Joseph, and Pohle, Frederick V. Line load applied along generators of thin-walled circular cylindrical shells of finite length. Quart. Appl. Math. 11, 411–425 (1954).

Bei der gegebenen Ableitung werden die Grundgleichungen für die Verschiebungen der dünnwandigen Kreiszylinderschale entsprechend der vereinfachten Annahme von L. H. Donnell [NACA Rep. no. **479** (1933)] angeschrieben. Ausser der Differentialgleichung achter Ordnung der einzigen abhängigen Veränderlichen w (Verschiebung in radialer Richtung) sind zwei Differentialgleichungen vierter Ordnung streng befriedigt. Die abhängigen Veränderlichen sind u und w in der einen Gleichung, v und w in der anderen, wo u die Verschiebung in Richtung x der Zylinderachse, v die Verschiebung in Richtung φ der Kreistangente bedeuten. Es sind Lösungen für die Verschiebungen, Biege- und Torsionsmomente, sowie Membrankräfte in geschlossener Form gegeben, hervorgerufen durch sinusförmig verteilte Linienlasten. Reihenentwicklungen dieser Lösungen sind geeignet, die Wirkungen abschnittsweise konstanter Belastungen längs der Erzeugenden sowie konzentrierter Einzellasten darzustellen. Die Frage der Konvergenz bei zunehmender Konzentration der Last wird am Schluss der Arbeit diskutiert. Solange die Belastung über eine endliche Strecke verteilt ist, konvergieren alle Ausdrücke für Verschiebungen, Momente und Spannungen, und ausserdem für eine konzentrierte Last ausserhalb des Angriffspunktes. Falls die Belastung ein konzentriertes Moment M_x in Achsenrichtung an der Stelle $x=0$, $\varphi=0$ darstellt, konvergieren für $\varphi=0$ die Verschiebungen und Membrankräfte aber die Momente M_x und M_φ (in axialer und tangentialer Richtung) sind durch divergente Reihen gegeben, die für $x\neq 0$ (Angriffsstelle des Moments M_x) null und für $x=0$ unendlich gross werden. Dagegen konvergieren alle Reihen für $\varphi\neq 0$. Wenn schliesslich die Belastung durch ein konzentriertes Tangentialmoment M_φ an der Stelle $\varphi=0$, $x=0$ gegeben ist, divergieren alle Reihen für die Momente für $\varphi=0$ als eine Folge der Darstellung der Belastung durch eine divergente Reihe. Dagegen sind alle Grössen durch konvergente Reihen gegeben, wenn φ von Null verschieden ist.

R. Gran Olsson (Trondheim).

Murnaghan, Francis D. On the effect of pretwisting on bending. Proc. Nat. Acad. Sci. U. S. A. **39**, 1218–1220 (1953).

The author shows that a long cantilever beam of circular section under the combined action of twisting and bending may readily have finite displacements of rotation and deflection near the free end and still suffer only infinitesimal strains. A first-order effect obtained by classical infinitesimal elasticity theory is that combined twisting and bending of such a circular section beam requires the existence of a transverse load of magnitude $k\mu c^2A/2R$ acting perpendicular to the plane of bending. Here k is angle of twist per unit length, μ is shear modulus, A is area of section, R is radius of curvature at origin and c=length along axes of beam.

D. L. Holl (Ames, Iowa).

Pister, Karl S. The Airy stress function in curvilinear coordinates with application to the uniform flexure of a naturally curved spiral beam. J. Franklin Inst. **257**, 25–36 (1954).

In problems of plane elasticity, in the absence of body forces, the stresses are derivable from a scalar function known as the Airy stress function. By expressing this relation as a tensor equation the use of the Airy function is generalised for any plane, orthogonal coordinate system. A general expression for the compatibility equation in terms of the stress function is given. It is found that considerable simplification results if isometric coordinates are used. The geometry of the isometric, curvilinear coordinate system is determined by a conformal mapping of the rectangular, Cartesian coordinate plane upon the curvilinear coordinate plane. For a given coordinate system the components of the metric tensor are expressible in terms of the mapping function.

As an application of the preceding theory the uniform flexure of a beam whose edges are bounded by logarithmic spirals is discussed. It is found that all spiral beams may be specified by two dimensionless shape parameters. Stress distributions in a typical beam are exhibited. As limiting cases of the solution for a spiral beam, the uniform flexure of a wedge acted upon by a moment at the vertex and the uniform flexure of a sector of a circular ring are obtained. (Author's abstract.)

R. M. Morris (Cardiff).

Sengupta, A. M. The effect of two equal circular holes on the stress distribution in a beam under uniform bending moment. Bull. Calcutta Math. Soc. **45**, 65–68 (1953).

Chandra Das, Sisir. On the stresses due to a small spherical inclusion in a uniform beam under constant bending moment. Bull. Calcutta Math. Soc. **45**, 55–63 (1953).

This paper deals with the stress concentration due to a "small" spherical inclusion situated on the neutral axis of a beam in pure bending. An exact solution in closed form is obtained for the idealized problem characterized by a spherical inclusion in an infinite medium for which $\tau_{zz}=Ax$ at infinity, while the remaining stress components vanish there. Here A is a constant and the inclusion is assumed centered at the origin. Numerical values are given corresponding to the rigid inclusion and in case the moduli of elasticity of the inclusion and the surrounding material are in the ratio one to four.

E. Sternberg (Chicago, Ill.).

Leggett, D. M. A. Summary of the theoretical work done on the stability of thin plates 1939 to 1946. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2784** (1950), 8 pp. (1953).

Neuber, H. Vereinfachung der Grundgleichungen der elastischen Stabilität mit Anwendung auf Stäbe, Platten und Schalen. Z. Angew. Math. Mech. **33**, 286–290 (1953).

In der vorliegenden Arbeit wird ein neuer Ansatz für den vom Verf. eingeführten [dieselbe Z. **23**, 321–330 (1943); diese Rev. **7**, 40] Tensor der Vorspannung angegeben, der die Integration der Grundgleichungen der elastischen Stabilität wesentlich vereinfacht, wobei sich die mit den Komponenten des Tensors der Vorspannung behafteten Glieder innerhalb der linearen Stabilitätstheorie gegenseitig aufheben, sodass sich die Bedingungen des Gleichgewichts auf das Verschwinden der Divergenz des durch die indifferente Formänderung geweckten (bzw. mit derselben durch das Gesetz der Elastizität verknüpften) Tensors der Zusatzspannungen einschränken. Sie gleichen daher den Differentialgleichungen der klassischen Elastizitätstheorie, bezw. den Bedingungen des Gleichgewichts des starren Körpers. Damit stehen die Verfahren der gewöhnlichen Elastizitätstheorie als Lösungen zur Verfügung, insbesondere im dreidimensionalen Fall der vom Verf. einge-

führte "Dreifunktionenansatz" [ibid. **14**, 203–212 (1934)]. Es wird gezeigt, dass der Tensor der Vorspannung lediglich bei der Aufstellung der Randbedingungen in Erscheinung tritt. Die Verwendbarkeit der Methode wird am Beispiel der Knickung prismatischer Stäbe nachgewiesen (Bestätigung der Eulerlast). Weiter wird im Fall der Plattenknickung nachgewiesen, dass sich die allgemeine Differentialgleichung mit Hilfe der neuen Formulierung unmittelbar aus der gewöhnlichen Plattentheorie ergibt, wobei sich Übereinstimmung mit bisherigen Formulierungen nach der energetischen Methode und als Variationsproblem herausstellt. Endlich lassen sich die Differentialgleichungen der Schalenknickung in relativ einfacher Weise aus der tensoriellen Schalentheorie herleiten [Neuber, ibid. **29**, 97–108, 142–146 (1949); diese Rev. **12**, 219], worauf Verf. noch näher eingehen wird. *R. Gran Olsson* (Trondheim).

Strasser, A. Zur Beulung versteifter Platten. Österreich. Ing.-Arch. **7**, 262–270 (1953).

Bei der Untersuchung der Stabilität der allseitig gelenkig gelagerten Rechteckplatte, die in ihrer Mittelebene durch linear veränderliche Randkräfte belastet und durch äquidistante Längs- und Quersteifen verstärkt ist, wird die ausgebogene Fläche angenähert durch eine Fouriersche Doppelreihe $w=\sum_m\sum_n A_{m,n}\sin(m\pi x/a)\sin(n\pi y/b)$ dargestellt. Bezeichnet A die Formänderungsarbeit, wird die Knickbedingung aus einem System linearer, homogener Gleichungen durch Nullsetzen von $\partial A/\partial A_{m,n}$ erhalten und ein allgemeines Bildungsgesetz der Knickbedingung in Matrixform für beliebige Anzahl der Steifen entwickelt. Weiter werden rekursive Näherungsformeln für die minimalen Steifigkeiten bei beliebiger Steifenzahl und konstanter Randbelastung angegeben. Über das Verhalten der Konvergenz der Näherungslösungen kann allgemein gesagt werden, dass mit zunehmender Anzahl der Steifen bereits die zweiten bis dritten Näherungen für praktische Zwecke vollkommen ausreichen. Die Konvergenz wird mit zunehmendem Seitenverhältnis $\alpha=a/b$ verbessert, aber bei $0<\alpha<2$ liefern erst die dritten bis fünften Näherungen hinreichend genaue Werte. *R. Gran Olsson.*

Milosavljević, Miodrag. Ein Beitrag zur Stabilität der gleichmässig gedrückten Rechteckplatte mit Steifenkreuz. Acad. Serbe Sci. Publ. Inst. Math. **5**, 109–114 (1953).

In dieser Arbeit wird die Knickbedingung für eine gedrückte dünne Platte, durch eine in beliebiger Lage rechtwinklige Aussteifung verstärkt, mit Hilfe der Differentialgleichung der ausgebeulten Platte aufgestellt, wobei die Annahme gemacht wird, dass die in der Druckrichtung verlaufende Steife ebenfalls Druckkräfte überträgt. Die Knickbedingung stimmt der von H. Fröhlich [Bauingenieur **18**, 673–682 (1937)] auf Grund der energetischen Methode aufgestellten Bedingung vollkommen überein. Das Problem wurde allgemeiner von A. Strasser in der oben referierten Arbeit gelöst. *R. Gran Olsson* (Trondheim).

Brown, E. H., and Hopkins, H. G. The initial buckling of a long and slightly bowed panel under combined shear and normal pressure. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2766** (1949), 19 pp. (1953).

As an approach to the general problem, the present paper gives a contribution to the stability problem under combined normal pressure and shear of a long and slightly bowed panel, which could be considered as a part of a circular cylinder of great radius, so that the present problem would appear to be covered by an earlier paper of the authors [same Rep. and Memoranda no. **2423** (1951); these Rev. **13**, 511]. However, the analysis by L. H. Donnell [NACA Rep. no. **479** (1933)] upon which the paper from 1951 is based, cannot easily be extended to take account of normal pressure for low degrees of curvature, and only the extension for fuselage or engine nacelle curvatures was attempted. On the other hand the present analysis is not readily applicable to high degrees of curvature, because the computation involved becomes excessive. The earlier and this paper together cover the whole range.

The theoretical values of the initial shear buckling stress agree well with the experimental values, despite the fact that the theory neglects the considerable inter-rib ballooning which must have occurred in the experiments. The critical shear stress increases slowly with curvature and sharply with normal pressure over the range considered. The wavelength of the buckles decreases with curvature but increases with pressure, but no experimental values are yet available for this. The difference between the buckling stresses for simply supported and clamped edges is considerable for a flat panel under shear alone but decreases rapidly with curvature and pressure, making the indeterminateness of real edge conditions less important. In the opinion of the authors future experimental work should consider the influence of normal pressure on the post-buckling behaviour of thin curved sheet under combined compression and shear stresses. This would be of great assistance in theoretical work on the combined loading case. *R. Gran Olsson.*

Mindlin, R. D., and Bleich, H. H. Response of an elastic cylindrical shell to a transverse, step shock wave. J. Appl. Mech. **20**, 189–195 (1953).

A modal analysis of dilatational, translational and inextensional-flexural modes is made of the interaction of an infinitely long elastic cylindrical shell of circular cross-section with its surrounding fluid medium when a plane shock wave parallel to the axis of the cylinder strikes the cylinder. The problem is treated as linear in that the response of the fluid medium is assumed to be acoustic and the boundary conditions are linearized. By means of an approximation which limits the solutions to the early stages of envelopment, it is possible to find explicit solutions for the displacement, velocity, acceleration and pressure at the shell associated with each mode. Typical curves are drawn for the dilational, translational and first flexural modes. For completeness, reference should be made to the work of G. F. Carrier [Grad. Div. Appl. Math., Brown Univ. Rep. no. **B11-4** (1951)] and M. C. Junger [J. Appl. Mech. **19**, 439–445 (1952)]. *P. Chiarulli* (Providence, R. I.).

Finzi, Leo. Proprietà delle strutture elastoplastiche nello spazio delle iperstatiche. Pont. Acad. Sci. Acta **15**, 121–136 (1953). (Latin summary)

Geometric discussions of the mechanical behavior of certain redundant elastic-plastic structures have been given by W. Prager [J. Aeronaut. Sci. **15**, 337–341 (1948); these Rev. **10**, 81], and P. S. Symonds and W. Prager [J. Appl. Mech. **17**, 315–323 (1950); Proc. Symposia Appl. Math. v. 3, McGraw-Hill, New York, 1950, pp. 187–197; these Rev. **11**, 560; **12**, 563]. For all but the simplest structures, the method employed by these authors to represent a state of stress and permanent strain by a pair of points ("stress

point" and "set point") requires the use of a Euclidean space of a rather large number of dimensions. The present author greatly simplifies the representation of the state of stress by using the fact that, for given loads, the state of stress is specified by the values of the redundants. For a truss, for instance, the earlier method requires a "stress space" whose number of dimensions equals the total number of bars, whereas the new method uses a "space of redundants" whose number of dimensions equals only the number of redundant bars. On the other hand, the influence of load variations on the forces and permanent elongations of the bars is more readily studied when the earlier method is used. In the present paper this problem is tackled only for structures with two redundants. For such structures a graphical method for determining the stresses and permanent strains is developed. *W. Prager* (Providence, R. I.).

Finzi, Leo. Strutture reticolari elastoplastiche: principio del minimo lavoro plastico. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 7–26 (1952).

Finzi, Leo. Sforzi e deformazioni nelle strutture reticolari elastoplastiche. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 16(85), 225–240 (1952).

The first paper discusses the mechanical behavior of an elastic-plastic truss with two redundant bars by means of geometric representations of the states of stress and permanent strain. The statement is made that there is a one-to-one correspondence between the states of loading on one hand and the states of stress and permanent strain on the other hand, provided that the loading process is such that no load ever decreases and no bar ever reverses the sense of its yielding. (The reviewer has not been able to follow the proof of the author or construct an independent proof.) In the second paper the author seems to concede (p. 227) that the example considered in the first paper is not of sufficient generality to permit such sweeping conclusions. In attempting to retrieve as much of the minimum principle as is possible, the author seems to be forced to consider the elastic-plastic behavior phase by phase (p. 232), a new phase beginning whenever a bar starts or stops yielding. As is well known, a minimum principle then holds for each phase. *W. Prager* (Providence, R. I.).

Hodge, P. G., Jr. The effect of strain hardening in an annular slab. J. Appl. Mech. 20, 530–536 (1953).

The author considers an annular slab of uniform thickness subjected to a uniform biaxial tension on the outer edge. It is assumed that the slab is in a state of generalized plane stress. The material of the slab is supposed to be elastic-plastic and obey Tresca's yield condition and the associated flow rule. During the plastic flow the yield hexagon remains centered at the origin and increases in size. The corresponding stress-strain curve in tension is approximated by a number of straight-line segments. Under these conditions the equations for the slow deformation of the slab are given and then solved by assuming that boundary motions may be neglected if the strains are small, and that elastic strain components may be neglected if the strains are large. *E. T. Onat* (Providence, R. I.).

Bishop, J. F. W. On the complete solution to problems of deformation of a plastic-rigid material. J. Mech. Phys. Solids 2, 43–53 (1953).

The recently developed theory of limit analysis has shed much needed light on the validity of the "solutions" usually considered in the Saint Venant-Mises theory of plasticity. These solutions consist of a velocity field, defined throughout the entire body under consideration, and a compatible stress field defined only in the plastically deforming part of this body. Such a "partial" solution, being kinematically admissible in the sense of limit analysis, furnishes an upper bound for the load intensity that first produces plastic flow in the plastic-rigid body. Only when the stress field is extended throughout the rigid part of the body so as to satisfy the equations of equilibrium without violating the yield inequality, is a "complete" solution obtained that furnishes the actual load at the yield point of the body. The present paper contains a discussion of the techniques of completing partial solutions. The fundamental theorems of limit analysis are reviewed. (These theorems, which the author attributes to R. Hill, were first obtained by Drucker, Greenberg, and Prager, in reports to which Hill refers in the paper cited by the author.) Since the yield inequality is difficult to handle mathematically when the possibilities of completing a partial solution are discussed, the author considers primarily such continuations of the partial stress field that represent a fully plastic stress state of the body. Theorems are established which, in certain typical situations, allow conclusions to be drawn regarding the existence or non-existence of a complete solution. For example, the following theorem is shown to be valid when the boundary conditions include fixed rigid constraints at which no relative motion occurs: If an incomplete solution exists to the actual problem, and a complete solution exists where all stress boundary conditions except those at the specified constraints are satisfied, then a complete solution exists for the actual boundary conditions. The role of stress discontinuities in the completion of partial solutions is investigated, and examples of curved stress discontinuities are given. Yielding of notched bars under conditions of plane stress and plane strain, and sheet extrusion are discussed as examples. *W. Prager* (Providence, R. I.).

Agababyan, E. H. Stresses in a tube under a sudden application of a load. Ukrain. Mat. Žurnal 5, 325–332 (1953). (Russian)

A circular tube of incompressible material is subject to a uniformly distributed internal pressure P beginning at time $t=0$, the external boundary being free. The author first finds the radial and tangential stresses in the elastic case, for $P=$const. Continuing, he allows P to be so great as to cause a plastic zone to develop, wherein $|\sigma_r-\sigma_\theta|=2k$. The time t^* at which the internal boundary reaches the plastic state and the boundary, $r=r^*(t)$, between elastic and plastic zones, are found for this case, and the change of the stresses and movement of the zone boundary are illustrated by sketches representing several successive values of t. A similar study is made for the case of loading by a pressure impulse at time $t=0$. *R. E. Gaskell* (Seattle, Wash.).

Tyabin, N. V., and Pudovkin, M. A. The flow of a viscous-plastic dispersive system in a conical diffusor. Doklady Akad. Nauk SSSR (N.S.) 92, 53–56 (1953). (Russian)

Les auteurs étudient l'écoulement d'un milieu dispersif, doué de viscosité plastique, dans un diffuseur conique. Le phénomène est régi par les équations écrites par Tyabin; celles-ci sont simplifiées, dans le cas particulier considéré, en utilisant les conclusions expérimentales de la thèse de Mme Lazovsky; en particulier les déplacements peuvent être considérés comme radiaux. Les auteurs tiennent compte

de ces faits pour former les expressions approchées des solutions des équations de Tyabin; les formules résolutives sont assez simples pour permettre une discussion détaillée de toutes les particularités du phénomène. Entre autres résultats, les auteurs donnent la loi du débit total en fonction de la pression; la relation qu'ils obtiennent est linéaire et paraît en bon accord avec l'expérience pour de grandes pressions.

J. Kravtchenko (Grenoble).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Blank, Albert A. The Luneburg theory of binocular visual space. J. Opt. Soc. Amer. **43**, 717–727 (1953).

Using a simplified set of axioms, the author presents the theory of binocular visual space developed by R. K. Luneburg [Mathematical analysis of binocular vision, Princeton, 1947; Courant Anniversary Volume, Interscience, New York, 1948, pp. 215–240; these Rev. **9**, 49, 735, 296], according to which this space is of constant negative curvature, a hypothesis supported by new experimental results here described. *J. L. Synge* (Dublin).

Flammer, Carson. The vector wave function solution of the diffraction of electromagnetic waves by circular disks and apertures. I. Oblate spheroidal vector wave functions. J. Appl. Phys. **24**, 1218–1223 (1953).

Solenoidal solutions of the vector wave equation

$$(\nabla^2+k^2)\mathbf{A}=0$$

are defined by $\mathbf{A}=\nabla\times\mathbf{a}\psi$, in which $\mathbf{a}$ is either a constant vector or the position vector and ψ a solution of the scalar wave equation. Oblate spheroidal vector wave functions are obtained if ψ is expanded in oblate spheroidal wave functions. The author gives explicit expressions for vector wave functions that are appropriate to diffraction problems in which the (y, z)-plane is the plane of incidence. Various expansions pertaining to plane polarized waves are derived.

C. J. Bouwkamp (Eindhoven).

Flammer, Carson. The vector wave function solution of the diffraction of electromagnetic waves by circular disks and apertures. II. The diffraction problems. J. Appl. Phys. **24**, 1224–1231 (1953).

The author obtains the rigorous solutions of the problems of diffraction of a plane polarized wave by a perfectly conducting circular disk and a circular hole in an infinite perfectly conducting plane screen for arbitrary direction of incidence. Two groups of vector functions defined previously [see the preceding review] are combined in such a way that both boundary and edge conditions are satisfied. The case of normal incidence is treated in detail. The results are checked against known results obtained by different methods. *C. J. Bouwkamp* (Eindhoven).

Agostinelli, Cataldo. Onde elettromagnetiche stazionarie in una cavità ellissoidale a tre assi con involucro metallico perfettamente conduttore. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **86**, 85–99 (1952).

The author considers steady-state electromagnetic oscillations in a perfectly conducting ellipsoidal cavity resonator. Following standard practice he separates the field components in ellipsoidal coordinates and obtains three differential equations of the second order and doubly-periodic coefficients. For a special choice of certain constants the author obtains by means of the Weierstrassian functions simple solutions which correspond to simple modes of the cavity. The solution the author offers does not lend itself to numerical evaluation. *C. H. Papas.*

Aymerich, Giuseppe. Un teorema di unicità sulle onde elettromagnetiche guidate da un guscio anisotropo. Boll. Un. Mat. Ital. (3) **8**, 273–276 (1953).

In this paper is proved a uniqueness theorem concerning the propagation of harmonic electromagnetic waves along a straight wave guide of arbitrary simply-connected cross-section. The material of which the guide is constructed is anisotropic in that, at any point, electricity can flow only in one direction, and this direction varies continuously from point to point. Such a guide is the mathematical idealisation of a non-conducting cylinder on which insulated perfectly conducting wire is wound. *E. T. Copson.*

Gans, Richard und Bemporad, Manuel. Zur Theorie der geradlinigen Antenne. Arch. Elektr. Übertragung **7**, 169–180 (1953).

The authors present an alternate method for the solution of the Hallén integral equation [Nova Acta Soc. Sci. Upsaliensis (4) **11**, no. 4 (1938), p. 11]. The steps lead to an infinite set of linear equations, through the substitution of a Fourier series for the unknown current. The method does not appear to be new [see, e.g., Aharoni, Antennae, Oxford, 1946, p. 160 [these Rev. **8**, 551] and the reference there to the book of Pidduck, Currents in aerials and high-frequency networks, Oxford, 1946; these Rev. **8**, 184]. In addition the authors appear unaware of the considerable body of work on the Hallén equation done by R. King and the group at the Cruft's Laboratory of Harvard University.

W. K. Saunders (Washington, D. C.).

Vilenskiĭ, I. M. On the theory of interaction of radio waves in the ionosphere. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 544–561 (1952). (Russian)

A radio station may produce a field sufficiently strong to affect the velocity of the charged particles in a portion of the ionosphere. This change in velocity changes the frequency of collisions between the charged particles and therefore, by the magneto-ionic theory, changes the absorptive properties of the ionosphere. If the signal from a second station passes through this disturbed portion of the ionosphere, it will receive any modulation present in the signal of the original station. This effect has been observed and is called the Luxembourg effect. Starting with some expressions from kinetic theory for the electron velocity distributions, the author uses physical arguments and an iteration method to obtain estimates of the magnitude of this cross-modulation. If ω_1 is the frequency of the original station, ω_2 the frequency of the second station, then the modulation of the original station will be received on the frequency ω_2 and also on the side-bands of frequency $\omega_2+2\omega_1$ and $\omega_2-2\omega_1$. The author also gives estimates of the magnitude of these latter effects. *B. Friedman* (New York, N. Y.).

Müller, Reinhard. Reflexion elektromagnetischer Wellen an inhomogenen Grenzflächen mit einer zum $\mathfrak{E}$-Vektor senkrechten periodischen Struktur. Arch. Elektr. Übertragung **7**, 492–500 (1953).

The complete solution of Maxwell's field equations is given for the case of a plane electromagnetic wave impinging

normally upon an infinitely thin plane layer made up of periodically alternating metal and dielectric strips. The reflected and refracted fields are computed, particularly for very narrow dielectric spacing strips so that simple approximations can be made. The solution is then modified for an arbitrary angle of incidence within a plane normal to the boundary layer and the results are applied to the reflection from an infinitely thin slit iris in a rectangular wave guide.

Finally, the author computes the field distribution for incidence of a plane wave upon an infinite stack of parallel metal plates; this solution is applied to the junction of two rectangular wave guides of different heights. A number of curves are given for auxiliary functions required in the solutions of the various problems. *E. Weber.*

Walker, L. R. The dispersion formula for plasma waves. J. Appl. Phys. **25**, 131–132 (1954).

Ott, H. Zum Energie-Impulstensor der Maxwell-Minkowskischen Elektrodynamik. Ann. Physik (6) **11**, 33–44 (1952).

The classical energy-momentum tensor of an electromagnetic field in the presence of a polarizable medium is made symmetrical by the addition of terms which then make the energy flow proportional to the momentum flow, as for the case of a field in vacuo. *H. C. Corben* (Pittsburgh, Pa.).

Emersleben, Otto. Das elektrostatische Selbstpotential äquidistanter Ladungen auf einer Kreislinie. Math. Nachr. **10**, 135–167 (1953).

$2n$ particles are arranged on a circle of radius r at equal distances from one another, and alternately carry charges ± 1. The self-energy of this arrangement is

$$ {}_{2n}^{0}\overline{\Phi} = \frac{n}{r} \sum_{k=1}^{(n)} (-1)^k \operatorname{cosec}\left(\frac{k\pi}{2n}\right), $$

where (n) indicates that only one half of the last term of the sum must be taken. The author analyzes ${}_{2n}^{0}\Phi$ in a manner similar to the analysis of the corresponding rectilinear arrangement [see Math. Nachr. **9**, 221–234 (1953) and the literature cited there; these Rev. **15**, 31]. The self-energy may be decomposed as ${}^{0}\overline{\Phi} = \overline{\Phi}_0 + \overline{\Phi}_l$, where

$$ {}_{2n}\overline{\Phi}_0 = -2n \sum_{k=1}^{(n)} \frac{(-1)^{k+1}}{k} $$

is somewhat similar to the potential of a rectilinear arrangement (except that the terms giving an "end effect" are missing), and $\overline{\Phi}_l$ is due to the curvature of the circular arrangement. The author then investigates the asymptotic behavior of the energy when n is large, distinguishing between even and odd values of n. The remainder terms are estimated. The corresponding problem for n equidistant and equal charges on a circle, with self-energy

$$ {}_{n}^{0}\Phi = \frac{n}{4r} \sum_{k=1}^{n-1} \operatorname{cosec}\left(\frac{\pi k}{n}\right), $$

is also discussed. Numerical values and/or graphs are given of ${}_{2r}\overline{\Phi}_0$ for $n=1(1)12$, ${}_{n}^{0}\Phi$ for $n=1(1)30$, $\sum_{k=1}^{(n)}(-1)^k k^s$ for $n=1(1)7$, $-2.6<s<3$. *A. Erdélyi* (Pasadena, Calif.).

***Marcuvitz, N. Network formulation of electromagnetic field problems.** Proceedings of the Symposium on Modern Network Synthesis, New York, April, 1952, pp. 215–235. Polytechnic Institute of Brooklyn, New York, N. Y., 1952.

The rigorous formulation of the network formalism at all frequencies and the proper use of impedance and scattering concepts rests upon the representation of the electromagnetic fields in an electrical structure by the superposition of an infinite number of modes. The author reviews the application to microwave structures, introducing characteristic vector mode functions for the transverse electric and magnetic field vectors, indicating their normalization and deducing the corresponding concepts of voltage and current as used in transmission-line theory. The concepts are illustrated for a vertical dipole above a planar dielectric structure with emphasis on the analogy to the treatment of circuit problems in terms of transient analysis by means of Laplace transforms. The final section outlines the general problems of synthesis of microwave networks and realizability conditions of the more generalized impedance functions encountered there. *E. Weber* (Brooklyn, N. Y.).

Quantum Mechanics

***Corson, E. M. Introduction to tensors, spinors, and relativistic wave-equations (relation structure).** Hafner Publishing Company, New York, 1953. xii+221 pp. $10.00.

HAVE

This excellent book is divided into essentially two parts: Mathematical Foundation and Physical Principles. Chapter I reviews briefly relevant topics in tensor analysis and elementary concepts in group theory. Chapter II deals with spinor analysis in terms of the 2- and 4-spinor formulations, the connection with world-tensors, and group theory. The presentation is based on the view that the theoretical physicist is primarily interested in the intelligent application of the rules of the spinor calculus rather than in the geometrical theory of the pure mathematician. Chapter III (the first of part II) develops from the canonical standpoint the essential field-theoretic concepts: the variational equations, the field-variable transformation properties, the conservation laws for mechanical and electrical entities—all of which stem from elementary requirements of invariance under wider and wider groups of transformations. Chapters IV and V, respectively, present the field or tensor-spinor and particle or matrix-algebraic aspects of relativistic wave-equations for particles of arbitrary spin and rest mass. The author pleads for an "all-particle theory" which contains all particles, or their representative wave-fields simultaneously, "because we know, experimentally, that there are transitions between nucleons and mesons; mesons, electrons and neutrinos; protons and photons, etc., so that seemingly any particle form can be obtained from any other, at least by intermediate steps". As is well known, W. Heisenberg is taking up the same ground in his recent publications. Even if the author emphasizes that he does not attempt to record "the long string of definitions and axioms, which may be found in any textbook on groups", nevertheless the regulating influence of group theory is evident everywhere (e.g., in Chapter II in connection with the isomorphism between the proper Lorentz group and the factor group $\mathfrak{S}_4/N$, where $\mathfrak{S}_4$ is the binary unimodular spinor group of linear transformations, and N the invariant subgroup consisting of the two elements e and $(-e)$). But why does the author avoid the concept "homomorphism" in dealing with the "isomorphism theorems"?). In his general field theory (Chapter III) the author starts from a variation principle which is simply an extension of Hamilton's principle of stationary action $\delta\int_{t_0}^{t_1}\mathfrak{L}dt=0$ with the Lagrangian $\mathfrak{L}$ now given by the square

integral of a Lagrangian density $\mathfrak{L}$. The derived field density tensors represent the energy momentum, angular momentum, etc. and their respective continuity equations which express the conservation laws. These results hold quite generally for arbitrary fields whether associated with charge or not, whose quantized form, therefore, describes charges or neutral quanta, respectively. In the case of complex fields, which are suitable for describing charged particles, the further condition of gauge invariance of first and second kinds permits the definition of a current-charge density 4-vector which satisfies the usual continuity equation. The whole theory can be cast in canonical Hamiltonian form in which the consideration of infinitesimal contact transformations as generators of the Lorentz group affords the proof of the Lorentz covariance of the formalism, and from which the canonical quantized theory of wave fields follows in the passage from classical to quantal Poisson brackets.

In the theory of relativistic wave equations the author starts with the following physical postulates: (A) the requirement of a unique rest-mass; (B) the requirement of a unique spin; (C) the requirement that either the total energy or the total charge of the field associated with the particle is positive definite. The Dirac-Fierz-Pauli theory satisfies all three postulates and its basic mathematical form is the second order equation, together with the necessary supplementary field equations on ψ's of appropriate transformation and symmetry character such that a given field manifests but one spin under all conditions.

However, the author, emphasizes that for spin >1 this theory cannot be cast in the simple Lagrangian form of the foregoing chapter III. He discusses also the Harish-Chandra theory. This theory satisfies postulates (A) and (C), but for spin >1 leads to what the author terms compound spin character, that is, the spin is not unique. Finally, the Bhabha theory rejects the first postulate (A) which is found to be equivalent to rejecting all three postulates for the general case of spin >1. All three theories agree in the cases corresponding to spins 0, 1 and $\frac{1}{2}$ (Dirac field, Proca field). In chapter V the author deals especially with representations of the inhomogeneous Lorentz group including the translations of the origin in space and time. The linear manifold of states $\psi_1, \psi_2, \cdots$ is transformed by the Lorentz transformation L into the states $L\psi_1, L\psi_2, \cdots$, which have in the new coordinate system the same properties as the $\psi_1, \psi_2, \cdots$ in the original system. The existence of such states is precisely the content of relativistic invariance. Since all Lorentz frames are equivalent for the description of the system, it follows that together with ψ, $D(L)\psi$ is also a possible state viewed from the original Lorentz frame (D being a linear unitary operator determined up to a unimodular factor by the physical content of the theory). The (invariant) manifold, therefore, contains with every ψ all transforms $D(L)\psi$, where L is any Lorentz transformation. The $L\psi_i$ can be expressed in terms of the ψ_i, the expansion coefficient defining elements of a (unitary) representation of the group, in general of infinite dimensionality, corresponding to the infinite number of possible wave functions.

At the end of chapter V the author discusses the invariant methods of E. Wigner which attack the problem of defining all relativistically invariant manifolds. Three general classes of fields are found: (1) quanta of discrete spins and non-zero rest-mass; (2) quanta of discrete spins and zero rest-mass; (3) quanta of continuous spin and zero rest-mass. *M. Pinl* (Dacca).

Husimi, Kôdi, and Ôtuka, Masuhiko. Miscellanea in elementary quantum mechanics. III. Progress Theoret. Physics **10**, 173–190 (1953).

[For parts I–II see Husimi, same journal **9**, 238–244, 381–402 (1953); these Rev. **14**, 1047; **15**, 79.] The ground state of a bound system may be determined by considering a suitable inequality, closely related to the uncertainty principle. Inequalities of the same general form (volume integrals of expressions bilinear in the wave function) may also be chosen to give excited states, and the application of these to perturbation theory is shown for some problems in one dimension. *H. C. Corben* (Pittsburgh, Pa.).

Infeld, L., and Plebanski, J. Electrodynamics without potentials. Acta Phys. Polonica **12**, 123–134 (1953). (Russian summary)

Electrodynamics is developed from a variation principle in which the Lagrangian is an arbitrary function of two invariants formed from the field strengths and their first derivatives. The theory is, in general, non-linear. Potentials do not play an essential role and neither, therefore, does gauge invariance. The first form of Dirac's new electrodynamics, the Proca and Maxwell fields are obtained as particular cases. *A. J. Coleman* (Toronto).

Suffczyński, Maciej. Note on electrodynamics without potentials. Acta Phys. Polonica **12**, 83–86 (1953). (Russian summary)

Applies the general methods of Dirac [Canadian J. Math. **2**, 129–148 (1950); these Rev. **13**, 306] to obtain the Hamiltonian formulation of a particular case of the potentialless electrodynamics of Infeld and Plebanski [see the preceding review] and is led to a theory precisely equivalent to Dirac's new electrodynamics [Proc. Roy. Soc. London. Ser. A. **209**, 291–296 (1951); these Rev. **13**, 893]. *A. J. Coleman.*

Infeld, L. On the use of an approximation method in Dirac's electrodynamics. Bull. Acad. Polon. Sci. Cl. III. **1**, 18–22 (1953).

The approximation method previously used by the author to study the equations of motion in General Relativity [Einstein and Infeld, Canadian J. Math. **1**, 209–241 (1949); these Rev. **11**, 59; Infeld and Wallace, Physical Rev. (2) **57**, 797–806 (1940); these Rev. **1**, 274] is applied to Dirac's new electrodynamics. The present note consists of general formal preparation which will be illustrated by specific examples in a future paper. *A. J. Coleman* (Toronto).

Rideau, Guy. Sur les principes variationnels en Mécanique quantique. C. R. Acad. Sci. Paris **237**, 1646–1648 (1953).

The author obtains an infinite number of equivalent variational principles of the type recently proposed by Kohn, Schwinger and Lippman, Blatt and Jackson, etc. for discussing quantum-mechanical scattering and shows that Schwinger's is the only one of these in which the expression to be varied is independent of the norms of the wave functions contained in it. *A. J. Coleman.*

Moshinsky, Marcos. Poles of the S matrix for resonance reactions. Physical Rev. (2) **91**, 984–985 (1953).

Tyablikov, S. V. Questions of invariance under translation in the theory of adiabatic approximation. Ukrain. Mat. Žurnal **5**, 152–158 (1953). (Russian)

A quantum-mechanical system with internal degrees of freedom is bound to a fixed centre by an external force. As

the strength of the external force tends to zero, the system will in the limit become free and acquire the property of being invariant under translations. The quantum-mechanical description of this limiting process is worked out in detail. The purpose of the paper seems to be mainly pedagogical. *F. J. Dyson* (Princeton, N. J.).

Wessel, Walter, and Czyzak, S. J. On the interpretation and generalization of Dirac's theory of the electron. Physical Rev. (2) **91**, 986–994 (1953).

The rest mass of an electron moving under Lorentz and radiation reaction forces is redefined so that it absorbs one of the usual two radiation reaction terms. The rest mass m satisfies the equation $m'=(2e^2/3c^2)u_k'u^{k'}$ where the dash denotes differentiation with respect to proper time along the world-line of the electron, and u^k is the unit tangent to the world-line. Simple, but arbitrary, assumptions lead to an expression for the angular momentum six-vector in terms of two invariants, u^k, and a new vector orthogonal to u^k. Poisson brackets for u^k, etc., can then be calculated and the way to quantization lies open. Assuming a relation, involving a hyperbolic cosine, between m and one of the above-mentioned invariants leads to a reasonable motion when the Lorentz force vanishes. However, if it does not, in order to predict motion approximating classical motion of the electron the Hamiltonian must be corrected by additional terms of such a nature as to elicit in the authors the hope that they will explain the Lamb-Retherford shift.

A. J. Coleman (Toronto, Ont.).

Costa de Beauregard, Olivier. Particule plongée dans un champ donné: définition des fonctions et valeurs propres au moyen d'intégrales quadruples. C. R. Acad. Sci. Paris **238**, 50–53 (1954).

Costa de Beauregard, Olivier. Superquantification de notre récent schéma. C. R. Acad. Sci. Paris **238**, 211–213 (1954).

Wiener, Norbert, and Siegel, Armand. Distributions quantiques dans l'espace différentiel pour les fonctions d'ondes dépendant du spin. C. R. Acad. Sci. Paris **237**, 1640–1642 (1953).

The authors have no difficulty in extending their work in Physical Rev. (2) **91**, 1551–1560 (1953) [these Rev. **15**, 273] to the case of spin-dependent wave functions.

I. E. Segal (New York, N. Y.).

Thermodynamics, Statistical Mechanics

***Samoĭlovič, A. G. Termodinamika i statističeskaya fizika. [Thermodynamics and statistical physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 439 pp. 12.70 rubles.

This book is intended as a somewhat elementary text for students of physics and mathematical physics. Table of contents: Fundamental principles of thermodynamics; fundamental principles of statistical physics; some applications of statistical physics to the investigation of classical systems; quantum statistics.

Kluitenberg, G. A., De Groot, S. R., and Mazur, P. Relativistic thermodynamics of irreversible processes. I. Heat conduction, diffusion, viscous flow and chemical reactions; formal part. Physica **19**, 689–704 (1953).

In this paper C. Eckart's theory [Physical Rev. (2) **58**, 919–924 (1940)] of the relativistic thermodynamics of irreversible processes in a simple fluid is extended to a mixture of n chemical components. Basic formulae are shown in Minkowskian coordinates ($x_4=ict$), so that their Lorentz invariance is evident. Each component j is characterised by a 4-vector $m_\alpha^{(j)}$ (product of proper density and 4-velocity), and a barycentric frame is defined by the 4-vector $m_\alpha=\sum_{j=1}^n m_\alpha^{(j)}$; u_α is a unit 4-vector in the direction of m_α. Vectors of relative flow of matter are defined by $I_\alpha^{(j)}=m_\alpha^{(j)}-c'^{(j)}m_\alpha$ $(j=1, 2, \cdots, n)$ to satisfy $I_\alpha^{(j)}u_\alpha=0$, and a heat flow vector $I_\alpha^{(0)}$ is introduced satisfying $I_\alpha^{(0)}u_\alpha=0$. The authors do not use the summation convention, but in this report repeated Greek suffixes are summed over 1, 2, 3, 4. The energy tensor has the form

$$W_{\alpha\beta}=u_\alpha u_\beta e'_{(v)}+c^{-1}(u_\beta I_\alpha^{(0)}+u_\alpha I_\beta^{(0)})+w_{\alpha\beta},$$

where $e'_{(v)}$ = energy per unit volume in the barycentric frame and $w_{\alpha\beta}$ is stress, satisfying $w_{\alpha\beta}u_\beta=0$. Viscosity, conservation of rest mass, momentum-energy law, first and second laws of thermodynamics, and entropy balance are discussed, and phenomenological equations are given in the form (*) $I_\alpha^{(j)}=\sum_{k=0}^n L_{\alpha\beta}^{(j)(k)}Y_\beta^{(k)}$ $(j=0, 1, \cdots, n)$, connecting the fluxes $I_\alpha^{(j)}$ and the forces $Y_\alpha^{(j)}$, the latter depending on external forces, temperature, proper densities, chemical potentials and u_α. Here $L_{\alpha\beta}^{(j)(k)}$ are $(n+1)^2$ tensors which, on the basis of isotropy, are expressed in the form $L_{\alpha\beta}^{(j)(k)}=(\delta_{\alpha\beta}+u_\alpha u_\alpha)L^{(j)(k)}$ $(j, k=0, 1, \cdots, n)$; the Onsager relations $L^{(j)(k)}=L^{(k)(j)}$ are assumed, and it is shown that of the $4(n+1)$ equations (*) only $3n$ are independent.

J. L. Synge (Dublin).

***de Groot, S. R. Hydrodynamics and thermodynamics.** Proceedings of Symposia in Applied Mathematics, vol. IV, Fluid dynamics, pp. 87–99. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. $7.00.

This paper is a useful summary of the views now current among specialists in "irreversible thermodynamics." [Most if not all of these ideas and many more are to be found in the works of Jaumann [Akad. Wiss. Wien. S.-B. IIa. **120**, 385–530 (1911); Denkschr. Akad. Wiss. Wien **95**, 461–562 (1918)] and Lohr [ibid. **93**, 339–421 (1917); Festschr. Deutsch. Tech. Hochschule Brünn, 1924, pp. 176–187].] The author speaks of a mixture of "components" when there are separate densities and velocities but only a common (total) stress, energy flux, entropy, etc. When these latter variables are introduced separately for each ingredient, the author speaks of a mixture of "fluids." [While he shows how to get the mean ("barycentric") velocity, he does not mention the possibility of defining mean or total stress, entropy, etc. for the "fluid" case.] The author does not assume the existence of an equation of state [cf. the reviewer, J. Rational Mech. Anal. **1**, 125–171, 173–300 (1952), see §32; these Rev. **13**, 794]. Instead, he takes its usual consequence, the differential equation for entropy, as a postulate. Thus internal energy, entropy, pressure, specific volume, and temperature all appear as primitive concepts in the author's theory, unrelated by the usual identities of classical thermodynamics. From the usual four conservation principles the author derives (for the "component" case) an entropy

equation of the form

$$\rho\frac{ds}{dt} = -\mathrm{div}\, \mathbf{J}_s + \sigma,$$

where $\mathbf{J}_s$ is a modified total energy flux vector and σ is a sum of the terms of the type "force times flux." The author's constitutive equations are all of the type "force" = linear combination of "flux". In a note added in proof he appears to realize that some of the statements in the text supposedly derived from considerations of invariance are false. [The reader may be disturbed also by the author's use of "theorem" to mean "assumption."] After pointing out the obvious fact that in a mixture it is impossible to get the velocities of the components without making use of the constitutive equations for the mixture, the author concludes: "the thermodynamics of irreversible processes is thus necessary for the justification of hydrodynamics . . . giving physical significance to [the stress tensor] etc." [The reviewer's conclusion from the same analysis is that writers in thermodynamics are just now beginning to use the phenomenological methods employed for more than a century in continuum mechanics; that they have so far limited themselves to linear models without time flux terms; and that they are not yet possessed of the algebraic and geometric tools which simplify such investigations.]

C. Truesdell (Bloomington, Ind.).

Popoff, Kyrille. Sur la thermodynamique des processus irréversibles dans le cas où la température et la pression restent constantes. C. R. Acad. Sci. Paris **237**, 698–700 (1953).

It is shown that the usual phenomenological relations are first integrals of the Newtonic system $d^2x_i/dt^2 = X_i$ in the case where the system exchanges heat with its environment, but its temperature and pressure remain constant.

C. C. Torrance (Monterey, Calif.).

Thomsen, John S. Logical relations among the principles of statistical mechanics and thermodynamics. Physical Rev. (2) **91**, 1263–1266 (1953).

Let p_i denote the probability that a given system with fixed energy E be in state i, where $1 \leq i \leq W$. Let λ_{ij} denote the transition probability per unit time from state i to state j. Make the debatable assumption that the definition $S = -k\sum p_i \ln p_i$ is applicable to irreversible phenomena. Consider the following propositions. M: $\lambda_{ij} = \lambda_{ji}$. D: $p_i\lambda_{ij} = p_j\lambda_{ji}$ at equilibrium. E: $p_i = p_j$ at equilibrium. S: $\dot{S} \geq 0$. L: $\sum_j \lambda_{ij} = \sum_j \lambda_{ji}$. It is shown that E, L, S are equivalent; that M is equivalent to D and E together; and that neither D nor E alone implies M.

C. C. Torrance.

Łopuszański, J. Distributions and statistical moments of bosons and fermions with some of their applications. Acta Phys. Polonica **11**, 298–313 (1953).

In the first nine pages the author derives, using generating functions, various formulas for probabilities connected with the random placement of N balls into n cells; the balls are either distinguishable or indistinguishable, and the random distributions may be subjected to conditions such as excluding multiple placements. In the second part the author surveys various special cases which have been treated recently in the physical literature and applies some formulae to the theory of photographic plates. [Most of the formulae attributed to modern physicists are classical and can be found in W. A. Whitworth, Choice and chance, 5th ed., Deighton Bell, Cambridge, 1901 reprinted 1942 by G. E. Stechert, New York.]

W. Feller (Princeton, N. J.).

Blanc-Lapierre, André. Sur l'application de la notion de fonction caractéristique à l'étude de certains problèmes de mécanique statistique. C. R. Acad. Sci. Paris **237**, 1635–1637 (1953).

Klein, G. et Prigogine, I. Sur la mécanique statistique des phénomènes irréversibles. III. Physica **19**, 1053–1071 (1953).

[For parts I–II see Physica **19**, 74–88, 89–100 (1953); these Rev. **14**, 829.] An infinite linear chain of identical particles with harmonic forces between nearest neighbors is considered and its evolution in time is studied by means of the explicit solution to the equations of motion first given by Schrödinger [Ann. Physik (4) **44**, 916–934 (1914)], for given statistical distributions of the initial conditions. Asymptotic convergence of each finite part of the chain to the equilibrium distribution is shown when positions and velocities are assumed uncorrelated initially. In terms of normal modes such uncorrelated initial distributions correspond to equipartition of energy among the normal modes, but nonrandom distribution of phases. Convergence toward the equilibrium distribution is thus clearly a consequence of the dispersion in the velocity of the plane wave vibrations. Although the abstract partly corrects this tendency, the paper derives all physical conclusions from mathematical properties of Bessel functions. In the reviewer's opinion, a treatment in terms of normal modes would be simpler and very much more transparent physically, and would lend itself in many instances to immediate extension to general 3-dimensional crystals. As an example, the equipartition theorem of section 4 becomes quite trivial in terms of normal modes, since potential energy and kinetic energy of a harmonic oscillator are well known to have equal average values.

The last part of the paper shows that the superposition principle is not able to give anything resembling the correct time-evolution of the distribution functions, and briefly considers the case of a finite chain.

L. Van Hove.

Falkenhagen, H., und Kelbg, G. Klassische Statistik unter Berücksichtigung des Raumbedarfs der Teilchen. Ann. Physik (6) **11**, 60–64 (1952).

M. Eigen und E. Wicke [Naturwissenschaften **38**, 453–454 (1951); **39**, 108–109 (1952)] haben unter Berücksichtigung des Raumbedarfs der Teilchen eine neue Verteilungsfunktion abgeleitet. Im folgenden wird diese Verteilungsfunktion auf beliebig viele Teilchensorten unterschiedlicher Grösse und somit verschiedener Besetzungszahlen erweitert. (From the authors' summary.)

C. C. Torrance.

Riddell, R. J., Jr., and Uhlenbeck, G. E. On the theory of the virial development of the equation of state of monoatomic gases. J. Chem. Phys. **21**, 2056–2064 (1953).

According to Mayer's theory [J. E. Mayer and M. G. Mayer, Statistical mechanics, Wiley, New York, 1940, chapter 13], occurrence of condensation for a gas depends on the convergence properties of the series expressing the pressure in powers of the activity, and of the virial series expressing it in powers of the density. The coefficients of

these series, cluster integrals and virial coefficients respectively, are sums of integrals, each of which is characterized by a linear graph composed of a number of points and of lines between these points. Study of the convergence requires an asymptotic evaluation of the number of graphs of various types and of the magnitude of the corresponding integrals. The present paper is devoted to the first part of the problem and solves it almost completely. It deals first with the counting of distinct graphs composed of distinguishable points, for disjoint and connected graphs and for the so-called stars. With the exception of stars, the counting is then performed for graphs of indistinguishable points. The paper ends with a number of remarks concerning the second part of the problem, i.e. the magnitude of the integrals. *L. Van Hove* (Princeton, N. J.).

Bernard, Jean-J. Application des distributions polynomiales à la détermination de l'épaisseur des ondes de choc. C. R. Acad. Sci. Paris **237**, 130–132 (1953).

In an earlier paper the author and R. Siestrunck [Recherche Aéronautique no. **31**, 45–48 (1953); these Rev. **14**, 1048] have given an extension of Mott-Smith's attempted kinetic theory of plane shock fronts [Physical Rev. (2) **82**, 885–892 (1951); these Rev. **12**, 891]. The author now proposes a second and somewhat different extension, in which the molecular distribution function is taken as a linear combination of p Maxwellian ones corresponding to different mean velocities in one direction and different temperatures. The coefficients are functions of the space coordinates only. The Boltzmann equation then yields a set of differential equations for the coefficients. He concludes that Mott-Smith's approximation ($p=2$) is the only one possible unless $p\geqq 5$. For the case $p=5$, the author gives a curve of shock thicknesses as a function of Mach number. Measuring the thickness in units of mean free path at the median of the wave, the author's result, for Maxwellian molecules, is that the curve of ϵ^{-1} against the oncoming Mach number M increases from 0 at $M=0$ to a maximum of about 0.15 when $M=$ about 2.7, falling off thereafter to 0 as $M\to\infty$. It is not possible to check details or even in all cases to follow the author's line of thought in this brief summary.
C. Truesdell (Bloomington, Ind.).

Landsberg, P. T. On Bose-Einstein condensation. Proc. Cambridge Philos. Soc. **50**, 65–76 (1954).

Various questions pertaining to the phenomenon of Bose-Einstein condensation for ideal gases are treated with more mathematical rigor than is customary. The proper limiting process of an indefinitely increasing volume with fixed chemical potential or density is carried out for a rather general form of the one-particle energy spectrum. The equilibrium distribution and the conditions for condensation are first derived for the grand canonical ensemble. It is then shown that the same results hold for the canonical ensemble. The proof of this important point makes use of an earlier and similar investigation by A. R. Fraser [Philos. Mag. (7) **42**, 156–164, 165–175 (1951); these Rev. **12**, 659]. Application of the saddle-point method to the ideal Bose-Einstein gas is finally discussed. *L. Van Hove.*

Dyson, Freeman J. The dynamics of a disordered linear chain. Physical Rev. (2) **92**, 1331–1338 (1953).

The author considers a linear chain of particles with harmonic forces between nearest neighbors. An entirely general method is given for the calculation of the spectrum of normal frequencies, proceeding through the calculation of the resolvent function of the eigenvalue problem involved and deducing the spectrum from the resolvent either by analytic continuation or by a double integral transform. The method is then applied to infinite chains with masses and forces randomly distributed according to two types of statistical law. A special case with a variable degree of order appearing as a parameter is treated in detail. The method applies also to a formally identical problem concerning transmission lines. *L. Van Hove* (Princeton, N. J.).

Mazur, P., and de Groot, S. R. On Onsager's relations in a magnetic field. Physica **19**, 961–970 (1953).

The Onsager reciprocal relations in the theory of irreversible thermodynamics are modified so as to include systems with a magnetic field which is a continuous function of space. If $L_{ji}(r', B'; r, B)$ is the coefficient in the phenomenological relationship between the jth generalized flux at the position r' in a magnetic field B at r' and the ith generalized force at the position r in a magnetic field B at r, then it is shown that the extension of the Onsager relations gives $L_{ji}(r', B'; r, B)=L_{ij}(r, -B; r', -B')$.

The consequences of these relationships are then considered as they apply to the electric and heat conduction tensors of an anisotropic medium in a magnetic field. A definition of an insulated system in which electromagnetic phenomena occur includes the energy of the fields external to the material system. It is found that the symmetric and antisymmetric parts of the electric conduction tensor are even and odd functions respectively of B. The symmetric part of the heat conduction tensor is also an even function of B but the properties of the antisymmetric part are somewhat more complicated since only the divergence of it has physical meaning. *G. Newell* (Providence, R. I.).

Mazur, P., and Nijboer, B. R. A. On the statistical mechanics of matter in an electromagnetic field. I. Derivation of the Maxwell equations from electron theory. Physica **19**, 971–986 (1953).

The macroscopic Maxwell equations for classical systems are derived from the microscopic equations of electron theory by a new method, based on ensemble averages and similar to Irving and Kirkwood's statistical derivation of the equations of hydrodynamics [J. Chem. Phys. **18**, 817–829 (1950); these Rev. **12**, 230]. Beyond introducing directly the physically relevant ensemble averages rather than space-time averages, the new method has the advantage of great simplicity and physical transparency. Whereas the usual derivations of the macroscopic Maxwell equations are restricted to systems at rest or systems moving with uniform velocity, the present derivation is general. It gives in particular a relationship between magnetic induction and field strength valid for arbitrary atomic motion.
L. Van Hove (Princeton, N. J.).

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